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ERRATA.

Page 69, lines 4 and 6 (from bottom); *for* "5" *read* "4."
—— 70, — 9 and 11; *for* "7" *read* "6."
—— 71, — 6 and 8; *for* "9" *read* "8."
—— 71, — 18 and 20; *for* "11" *read* "10."
—— 73, — 2 and 4; *for* "13" *read* "12."
—— 73, — 17 and 18; *for* "15" *read* "14."
—— 76, line 13; *for* "their" *read* "these."
—— 84, — 14 (from bottom); *for* "then" *read* "there."
—— 89, — 10; *for* "worked" *read* "marked."
—— 132, — 1 of § 4; *for* "(1)" *read* "the general quintic."
—— 140, — 7 of § 27; *omit* "-5" *and* "$+K$."
—— 140, — 9 of § 27; *omit* "and K is known."

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OF MANCHESTER.

I. — *On the Composition and Derivation of Rosolic Acid.* *By* R. Angus Smith, Ph. D., F.R.S.

Read December 1st, 1857.

When engaged in making an examination of ammoniacal liquor from the Manchester Gas Works, I found that a solution, from which the sulphur united with the ammonium had been removed, became, after a week or two, of a very beautiful rose colour. The amount of substance necessary to produce this colour in a portion of liquid contained in a beaker glass, was extremely small, scarcely capable of being seen, much less of being examined carefully. To have made use of portions of the gas water sufficiently large to obtain enough for examination would have required operations of very inconvenient magnitude.

Some time after this, in 1855, Mr. Mc. Dougall found that the lime which was exposed to the currents of certain vapours in his works became red. These vapours arose

from the operation of manufacturing the disinfecting powder called by his name, where carbolic acid or kreasote from coal is used. When he observed this red colour he renewed the surface of the lime and obtained a fresh portion. By continual renewals in this manner he obtained a crust of a very deep colour. This he brought to me. There were several pounds weight of it; but although the colour appeared so deep, a very small portion of colouring matter was found.

When this lime compound was heated with alkalies, a portion of the colouring matter was removed. I used ammonia, but soda or potash answered the same purpose. It was, however, not possible, or at least it was exceedingly tedious, to remove the whole in this manner, and when the lime was dissolved by muriatic acid, a black resin was found first swimming in a melted state in the rather warm liquid, and afterwards sinking and cooling, a dark resinous substance. The power of giving colour was remarkable, as I did not obtain more of this resin than about an ounce to a hundredweight of the lime compound. Of course the colour in these cases led me to think of Runge's rosolic acid, which he has described in 1834 in Poggendorff's Annalen (vol. xxxi. p. 70) in the following manner:

"Rosolic acid, or rosaöleic acid (Rosolsäure oder Rosaölsäure), is a product of the chemical decomposition of coal oil, and is so much the more remarkable on account of acting like a true pigment. With a suitable mordant it gives colours and lakes before which those of safflower, cochineal and madder must give way.

"Rosolic acid is a resinous mass, which may be reduced to a powder; it possesses a beautiful orange colour. I have not been able to find in the coal oils the substance from which it is produced; but to show its existence it is only necessary to shake the oil with milk of lime, to filter, and to boil for a few hours. The solution, at first colour-

less, becomes of a dark red colour. This colour arises from the rosolate of lime which, on standing, is precipitated as a highly coloured red powder."

From this it appears that Runge did not find its derivation, although it has generally been put as an appendix to carbolic acid.

In separating this resin from the lime by means of very dilute muriatic acid, there was at first a separation of fine light brown flakes; these at last gradually fell and united at the bottom of the vessel into a resinous mass. When this was dissolved in alcohol or ether, the solution had a deep brown colour, but no appearance of the rose colour; neither had it in any stage the beautiful orange colour mentioned by Runge. It is a question, therefore, whether it is right to give it the same name; but in many particulars it is so like it that I have made no alteration. When treated with alkalies these shades of brown, or reddish-brown, gave way before the peculiar rose colour. The finest colour in solution appeared to me to be that with ammonia; in a solid state, the finest was certainly that of the lime compound.

In order to find exactly its origin, I purified some carbolic acid made from coal by heating it with caustic soda, decomposing with acid and distilling. A vessel of lime was suspended over this under a bell-jar for some days, and gradually a rosy colour appeared on the surface. Another portion of carbolic acid was mixed with the lime and allowed to stand: this became of a very dark brown, and although rosolic acid was formed it was mixed with other substances which made it appear very impure. I then filled a glass tube with caustic lime and passed the vapour of the acid over it, allowing free access of air, heating one end of the tube nearly to redness and allowing the heat gradually to decrease to the other end, so as to observe at what temperature the formation of the

resin took place. Even after several hours no colour was observed. This tube was kept for some days, and along the whole of it the rose colour appeared, gradually deepening for some weeks.

As it seemed from these experiments to be produced by the oxidation of carbolic acid, and as the above method was tedious, I heated carbolic acid with caustic soda and black oxide of manganese so highly that the carbolic acid began to evaporate. In this manner it was formed with great rapidity; the melted mass became of a fine iridescent appearance; it was dissolved in water, and after a time, the manganate of soda having expended its oxygen, the oxide of manganese fell and the rosolic acid remained with the soda. The rosolate of soda was then decomposed with acid; a large quantity may in this way be procured with very little trouble.

The hypochlorites also cause the formation of a rosolate, but the process did not seem to promise so well as the one given.

To ascertain its composition, the resin obtained from the lead-salt was dried in a water-bath: it then becomes brittle, as Runge states, and may be powdered, but much drying destroys it. Many analyses, made at various stages of the drying, and too tedious and uninteresting to enumerate, showed a gradual rise of carbon in the composition and also a loss of hydrogen. The following analyses were made with chromate of lead and a stream of oxygen. Grains are used as weights:

	Substance.	Carbonic acid.	Water.
1.	3.017	7.802	1.661
2.	3.007	7.795	1.528
3.	3.861	10.087	2.029
4.	3.077	7.983	1.565

These are by per centage:

	I.	II.	III.	IV.
Carbon	70.527	70.698	71.256	70.875
Hydrogen.....	6.116	5.646	5.839	5.651

This corresponds to a composition of —

	Equivalents.	By calculation.	Mean of experiments.
Carbon.....	12	70.588	70.837
Hydrogen	6	5.882	5.814
Oxygen	3	23.530	23.349
		100.000	100.000

The true formula is perhaps C_{24} H_{12} O_6. The formula of carbolic acid is C_{12} H_6 O_2. Rosolic acid appears thus to be simply a result of the oxidation of carbolic acid. A compound of a very different nature has been got from catechu, called oxyphenic acid: it is C_{12} H_6 O_4, and is considered to be a higher oxide of the radical phenyle.

Having frequently prepared rosolic acid, I was desirous of obtaining the salts, and formed many compounds of it with lead. These compounds are brown and amorphous; they were obtained by precipitating an alcoholic solution of rosolate of potash with acetate of lead. I was not, however, satisfied with the results obtained by analysis. A very indefinite amount of lead is taken down, and although on three occasions I obtained a salt which was composed of nearly three of acid to two of lead, it was not exactly so. Another time it consisted of two of acid and one of lead. On one occasion the hydrogen was to the carbon in the proportion of 5:12, and on another it was 4:12, showing, I believe, one and two atoms of water removed. It may, however, be a long time before I can give any attention to the salts. None of them except perhaps the lime salt, as Runge states, showed any crystalline structure.

When treated with nitric acid it produces, like carbolic acid, picric or trinitro-carbolic acid (C_{12} N_3 H_3 O_{14}). By acting on it with chlorate of potash and muriatic acid, I did not obtain chloranil, but a greyish resinous substance. If the ammoniacal solution containing an excess of ammonia be allowed to stand for some months, it is

decomposed; a blueish scum appears on the liquid, and the sides of the vessel are covered with fine crystalline needles which have a blueish appearance. This substance I have only a very small portion of, but may obtain more for a future occasion.

If a colour should be found identical with Runge's, I should call this acid *pic-erythric acid*. The colour which this substance always assumes appears to establish an essential difference between it and the acid obtained by Runge, but its action with bases is still more widely apart. Runge says that it unites with mordants. I have thrown it down with oxides of tin and iron, alumina and other bases, but have found that it has by no means a power of retaining hold of the base. The colour of these compounds on cotton is certainly very beautiful for a short time, but carbonic acid or a weak alkali removes it rapidly. It could never, therefore, be used as Runge's substance to form a dye, or it might be obtained in very great abundance, and at a very cheap rate, in all gas-consuming towns. Another reason why it could not be used as such is that its peculiar rosy colour is only developed by alkalies; the alkalinity being removed, it is a dull resin again. This occurs very readily, the carbonic acid of the air being sufficient for the purpose. The magnificent colour of the lime compound is therefore very fugitive. If used to dye silk or wool, it imparts, if in an alkaline state, the same magnificent colour as to cotton, but the same evanescence is soon proved, and the cloth appears of a dirty brownish hue.

Runge says also that it can be used as a pigment. But of itself it has no beauty. Its salts also fail as a pigment, from the reasons above given. The lime salt, for example, which would be the most beautiful for the purpose, must be kept strongly alkaline, as the carbonic acid of the air readily reproduces the dark and dirty brown. I consider

that hereby one of the delusions regarding colours and other riches from coals is destroyed.

Another resinous body of a pure red, when alkaline, without any shade of rose, is found along with it, and several new substances will certainly come out of this field if laboured in.

The scientific examination of this substance is by no means completed here; but for the first time a mode of obtaining it at pleasure is put into the hands of chemists.

II. — *On the Consumption of Coal in Furnaces, and the Rate of Evaporation from Engine Boilers.* *By* JOHN GRAHAM, ESQ.

Read February 23rd, 1858.

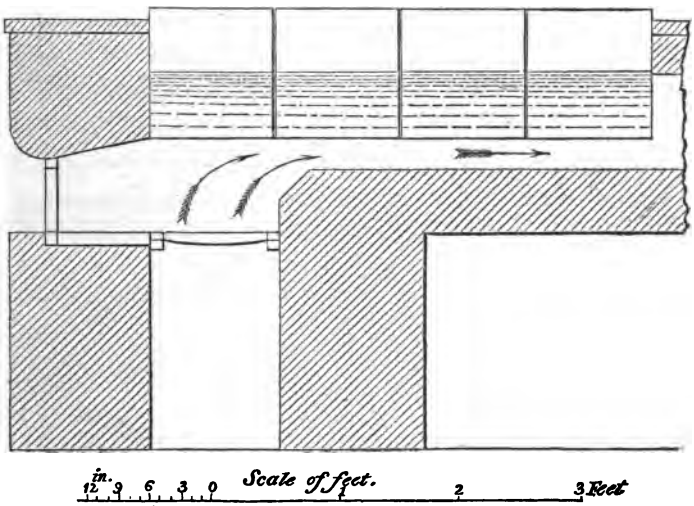
THE experiments to be detailed in the present paper were originally undertaken with a view to beneficial application, and were conducted, as they are now described, in as simple a manner as I was capable of doing. I have been desirous to avoid all entanglements arising from the use of formulæ and the introduction of theoretical speculations; the great and important field of practical knowledge to which the enquiry belongs being as yet, in my humble opinion, too little explored to admit of much generalizing; and I have aspired only to lay down a road into the region, hard and dry, on the basis of sure experiment.

I. SERIES OF EXPERIMENTS.

The experiments which I shall first describe were made with the view to ascertain the comparative evaporative power of equal but differently placed areas of heating surface, for the purpose of deciding whether we should look most, as regards the saving of coals, to the improvement of the fire-place, or the extension of evaporating flue surface. The results obtained were in accordance with general observation. Four open tin pans, each twelve inches square, were carefully fitted up in brickwork, and arranged together side by side, as seen in *Fig.* 1, the flame-

bed or flue being made to communicate with the main flue of a factory, to insure the draught being under control.

Fig. 1.



The evaporation from the pan first in order represents the direct heating effect of the fire; the evaporation from the second may be supposed to represent the heating effect of an equal surface of blaze; the evaporation from the third and fourth the effect of heated air only.

With a moderately strong draught the average rate of evaporation from the four open pans was as follows:

From the first pan,	100	67.6
From the second „	27	18.2
From the third „	13	8.8
From the fourth „	8	5.4
		100.0

Care should be taken to compare in each case the *scale* attached to the drawing.

II. Series of Experiments (small boilers).

Fig. 2.

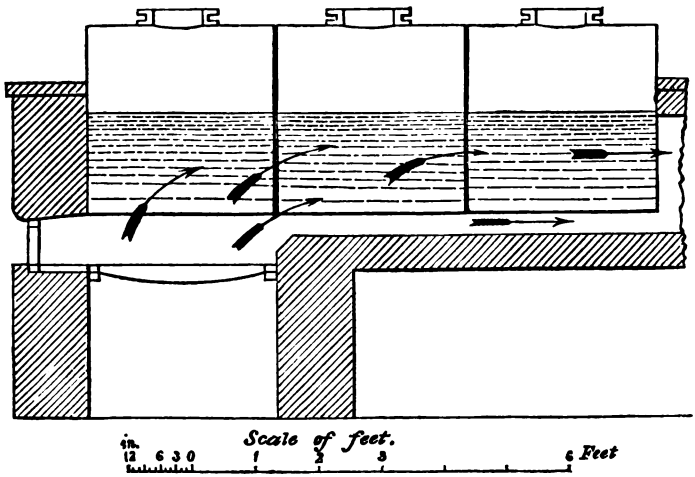
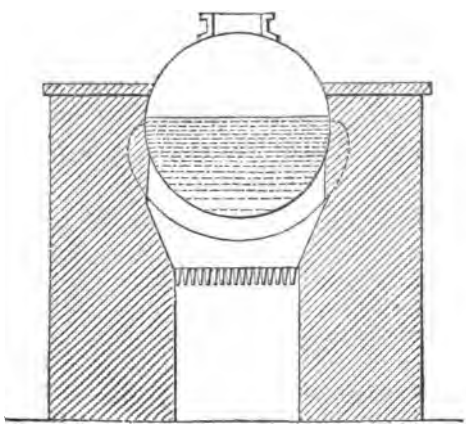


Fig. 3.



A second set of experiments was made with a similar object. The evaporators now employed were three separate plain cylindrical boilers, each three feet in diameter and three feet in length, open to the atmosphere, and placed close behind each other, being well set in brickwork.

The fire-place under the first boiler was two feet wide by three deep.

The fire-bars were half an inch thick, with half inch spaces.

The distance of the fire-bars from the bottom of the boiler was nine and a half inches.

The distance of the flame-bed from the bottom and sides of the second and third boilers was four inches.

The boiler plate was of quarter inch iron.

The comparative evaporating power of the three boilers so placed, was found to be as follows:

Table A.

Number of Experiment.	Duration of Experiment.		Coals Consumed.	Comparative Evaporation, the Evaporation from the first Boiler being 100.			Total Water heated from 60°, and evaporated from the three Boilers by one pound of Coal.
	Hours	Min.	Lbs.	1st Boiler.	2nd Boiler.	3rd Boiler.	Lbs.
1 *							
2	6	30	420	100	22.8	13.8	5.26
3	10	55	811	100	27.8	15.1	5.19
4	6	30	694	100	39.4	20.3	4.68
5	9	10	480	100	33.5	16.4	3.98
6	6	50	500	100	32.9	12.8	4.42
7	8	15	473	100	29.0	12.0	4.87
8	9	25	473½	100	28.6	9.7	3.83
9	5	15	489½	100	37.0	16.0	4.63
10	7	00	484	100	36.0	13.0	4.02
11	5	25	486	100	50.7	24.0	4.64
12	7	45	487	100	44.4	22.9	4.59
11	83	00	5798	1.100	382.1	176.0	50.11
Average	7	22	527	100	34.7	16.0	4.55

Evaporation from the first boiler, 100.0 ..... 66.4
„ from the second „ 34.7 ..... 23.0
„ from the third „ 16.0 ..... 19.6
100.0

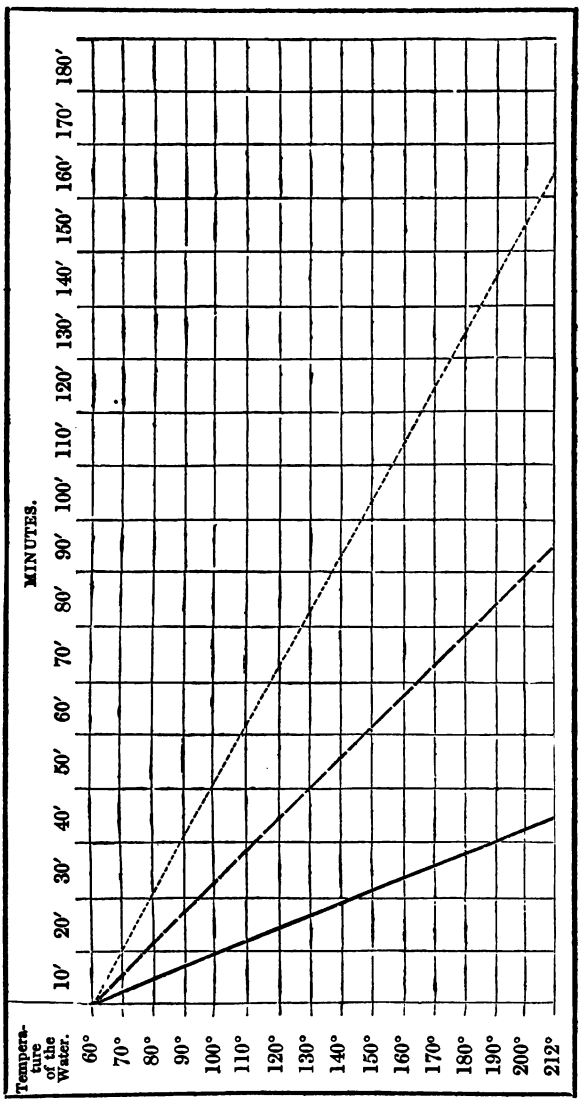
The ultimate result for the three boilers together, calculating the evaporation from 212°, is 5.31 lbs. of water evaporated from 212° by one pound of Worsley coal.

* A preliminary experiment omitted.

The rate of heating the water in the series of three boilers, placed as described, was also observed:

The results are expressed by the lines in the following diagram, *Table B*, showing the average rate at which the

Table B.



three small boilers placed in series were heated from 65° to 212°; the rate of the first boiler being represented by the line ———, the rate of the second boiler by – – – –, and the rate of the third by ----------:

To be heated from 65° to 212°, the first boiler required 40 minutes, the second boiler required 92 minutes, the third boiler required 161 minutes; or the heat acquired in equal times by the three boilers in series was by the first boiler 100°, by the second boiler 43.5°, by the third boiler 24.9°.

By comparing the amount evaporated from the third boiler, *Table A*, with its comparative rate of heating as above, it will be seen that flue space acts much more usefully in heating cold water up to 212° than in boiling it off, namely in the proportion of 24.9 to 16, the third boiler in both cases being considered as flue surface only.

The progressive manner in which heat is communicated, when the temperature of the body to be heated is not greatly under that of the heating body, is illustrated practically for steam as the vehicle of the heat by the following Table. The apparatus used was a square wooden cistern with an iron bottom, double-cased for the purpose of admitting steam. The cistern being filled with water at 60°, steam of the temperature of 218° was admitted.

Table C.

Time in Minutes.	Temperature of Water in Cistern.	Increase of Temperature.
0′	60°	0°
10′	100°	40°
20′	134°	34°
30′	158°	24°
40′	174°	16°
50′	183°	9°
60′	192°	9°
70′	198°	6°
80′	201°	3°
90′	206°	5°
100′	210°	4°

In experiments of this nature it is difficult to raise the water in the cistern beyond 210°. The increasing heating activity of the steam towards the end, as shown by the above Table, may possibly be accounted for by the increased motion in the water as it came near the boiling point.

Although the experiments were carefully performed with these small boilers, the results present a want of harmony among themselves, as regards the amount of water evaporated, and are not in accordance with the results obtained with other boilers, also of small dimensions. The effects produced by disturbing causes seem to be much more considerable when the boilers are small. At the risk of being tedious, I subjoin a notice of such disturbing causes as were recognized during the performance of the small boiler experiments (*Table A.*) Some of these influences appear of an unimportant character, and really are so in large experiments; but it would appear that slight disturbing causes may with small boilers produce greater deviations than is generally believed. It will also be observed that uniformity is not always attained even when the disturbing causes, so far as known, appear to be nearly equal. And lastly, that such causes appear occasionally to give the opposite result with small boilers from what they give with boilers with larger fire-places.

This discrepancy in results obtained in the "small way," as compared with those obtained on the "large scale," is well known to men of practical skill, and constitutes our chief difficulty in subduing science to our requirements. The case, for example, of dyeing "patches," as compared with dyeing "pieces," may be mentioned as one where disturbing causes rise to an extreme.

Disturbing causes recognized during the performance of small boiler experiments in Table A.

The First Experiment was rejected in consequence of the brickwork setting of the furnace being cold and damp.

Second Experiment. — Barometer 29.4 inches; hydrometer dry bulb 69°, wet bulb 67°; weather cloudy, with showers; draught strong; fired every 15 minutes; feed water 70°.

Third Experiment. — Barometer 30 inches; hydrometer 72° dry, 70° wet; weather dry and clear; draught good; fired every 10 minutes; feed water 70°.

Fourth Experiment. — Barometer 29.8 inches; hydrometer 70° dry and 64° wet; heavy rain; draught good; fired every 40 minutes; the fire-place was nearly filled each time with coals; feed water 70°.

Fifth Experiment. — Barometer 29.9 inches; hydrometer 66° dry and 60° wet; draught good; fired every 15 minutes; the burning fuel was kept about 3 inches deep only. In consequence of the coals being of the usual irregular size, some parts of the bars became bare occasionally. Feed water 70°.

Sixth Experiment. — Draught strong; thickness of fire 4 inches; feed water 60°. In other respects the same as fifth experiment.

Seventh Experiment. — Draught slow; thickness of fire 3 inches; feed water 60°. In other respects the same as fifth experiment.

Eighth Experiment. — Draught slower still; thickness of fire 3 inches; feed water 65°. In other respects the same as fifth experiment.

Ninth Experiment. — A repetition of the sixth experiment.

Tenth Experiment. — A repetition of the seventh experiment.

Eleventh Experiment.—Draught good; thickness of fire 5 inches. The coals in this experiment were fed upon the inner end of the fire-bars by a hole in the side of the brickwork, the red hot coals being pushed forward towards the front by the rake.

Twelfth Experiment.—A repetition of the eleventh experiment, but with a slower draught and a thinner fire.

III. Series of Experiments (small boilers.)

Fig. 4.

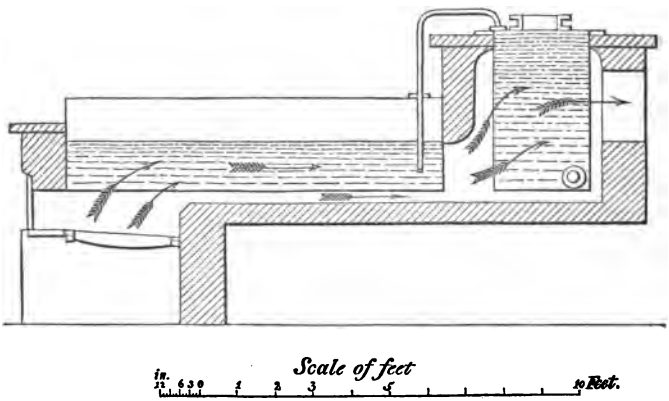
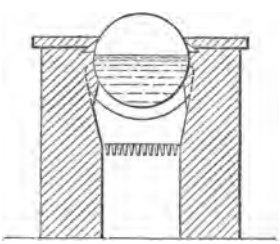


Fig. 5.



The next set of experiments was also made to ascertain the value of flue space, but more attention was now given to the capacity of the hot air after it had passed from under the boiler to heat a separate boiler. The boiler used

in this case was similar in construction and setting to the previous three small boilers, but was 10 feet long. The supplementary boiler was 4 feet 6 inches long, standing on end as seen in the drawing. It was worked for a lengthened period to heat the feed water made to pass through it, with the constant result that the feed water was heated variously from 170° to 180°.

The draught went directly under the main boiler, and was made to impinge against the side of the supplementary boiler and to pass round it and under it. Where the draught so impinged, the surface remained free from soot.

The average result was as follows, allowing in the calculations one increment of heat to raise the temperature of the water from 60° to 212°, and five increments to boil it off:

Heat acquired by principal or working boiler, 100...	87
„ supplementary „ 15...	13
	100

It appears from a careful set of experiments made in this neighbourhood, under the inspection of another observer, that when means are employed to keep the supplementary heating surface free from soot, a still more favourable result is obtained, viz:

Heat acquired by water in working boilers, 100	78.7
„ by water in heating pipes, 27.4....	21.3
	100.0

In the last experiments upwards of one-third of a mile of 4-inch piping was in use, which received the escaped heat from six large boilers, each 42 feet long. Six boilers, assisted by the supplementary heating surface, were found in practice to do the work which had previously required eight boilers, without such assistance. It should be noticed that in neither case was the principal boiler giving more than 7.5 pounds of water for the pound of coal

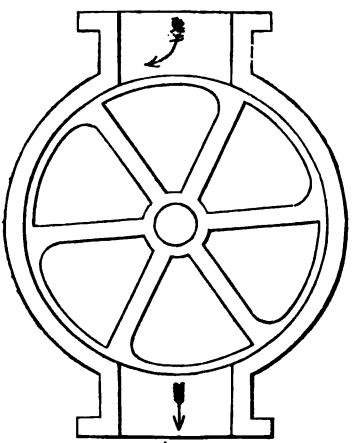
consumed. I have no results to show what the effect might be if the supplementary heating apparatus had been applied to a boiler capable of yielding itself a larger result.

IV. Series of Experiments.

The next set of experiments was made for the purpose of ascertaining the evaporative power of engine boilers of various shapes and of large dimensions.

In order to do this I found it necessary to contrive a simple water meter capable of registering the supply of water to the boiler with accuracy. This machine is exhibited in the accompanying figure.

Fig. 6. *Water Meter.*



This meter worked with perfect accuracy for several years, as was ascertained by occasionally checking its registered indications by means of a water cistern of known capacity attached. The registering apparatus was similar in construction to that used in gas meters, and was attached to one end of the axle. The interior part of this apparatus, which conveyed the water into the boiler, was not unlike a large "greasing tap." It was driven by a fast

and loose pulley, the direction of which was entrusted to the stoker, while the levels of the water in the boiler, and other circumstances at the beginning and end of each experiment were registered by another person. The drawing scarcely gives a proper idea of the mode by which the water was conveyed round and into the boiler.

I shall now give an account of the results obtained by me with the four different shapes of boilers in most common use in this neighbourhood. I should remark that before beginning to register results, the boilers in each case were re-set, and placed, by careful and continuous experiments, into what was found, to the best of my judgment, to be their condition for giving the best working result. The boilers were, further, always amply protected by coverings from loss of heat by radiation. The experiments of the present series were each of twelve hours duration. They commenced at 6 a.m. and terminated at 6 p.m., except those performed on Saturdays, which lasted from 6 a.m. to 1 p.m. It was also the practice to commence and end with the steam at, as nearly as possible, 7 lbs. of pressure.

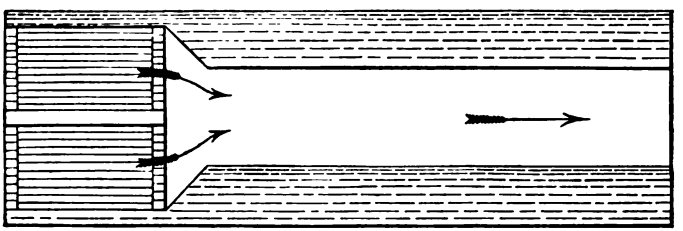
As each experiment of importance was carried on day by day for four or six weeks, and often repeated under varying circumstances at different times of the year, also with various boilers, with coals of various kinds, and different stokers, the experiments altogether occupied the better part of several years. At first the disturbing circumstances were found very perplexing, as they led to variations in the results which at the time could not be explained; but in proportion as these circumstances became known and were reduced in number, it became possible to proceed with more satisfaction and precision. The coals used in these experiments were the usual Worsley qualities, except in the case of the experiments made with the cylindrical boiler, described under the head IX. Series of Experiments, where Dukinfield coal was used. I had, however,

previously and have since ascertained that the Dukinfield and Worsley coals gave the same result.

Command was always had of the draught. The temperature in the flue at the bottom of the chimney, during some periods of the day, was sufficient to melt lead but never zinc.

V. Series of Experiments.

Fig. 7.



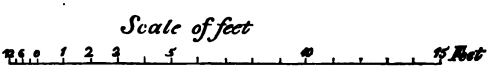
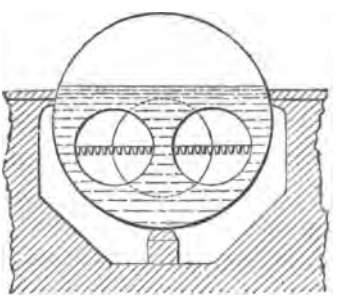


Fig. 8.



The boiler first experimented with was of the shape shown in the annexed drawing, and commonly called in this neighbourhood the "breeches boiler." It has two interior fire-places which join behind the bridges and form one direct flue through the interior of the boiler. It was 23 feet 3 inches long, 8 feet in diameter.

Fire-places each 3 feet by 6 feet, giving 18 square feet of fire surface.

Internal flue 3 feet 9 inches in diameter.

Fire-bars half inch thick with half inch spaces.

The boiler plate $\frac{3}{8}^{\text{ths}}$ of an inch in thickness.

The experiments made with this boiler were continued under different circumstances and at various times of the year, and the average result obtained, working under the best conditions I could command, was 5.90 lbs. of water evaporated from the temperature of 84° by one pound of coal consumed.

Or if we add one-sixth to this result as equivalent to the heating of the feed water, the ultimate result will stand thus:

In the breeches boiler 6.88 lbs. of water are evaporated from 212° by one pound of coal.

In all the experiments made with this boiler, though my results were affected more or less by what was considered to be a slightly deficient draught, still each fire evaporated 24.429 lbs. of water from 60° in the course of twelve successive working hours, and generally throughout the series it was found that this boiler was more affected by disturbing causes than the boilers of other shapes, possibly in consequence of the comparatively small size of the fire-places.

The smallness of the yield from this boiler is a matter of surprise. I have, however, no reasonable grounds for suspecting the accuracy of my experiments, since they were so often repeated and under such varied circumstances.

VI. Series of Experiments.

I may add that in a short series of experiments made on two other boilers of an analogous construction, namely, with two fire-places and two internal flues passing from

end to end, (a construction of boiler which greatly recommends itself for strength, durability, compactness, and simplicity of form,) working with an excellent draught, and under a pressure of 57 lbs. on the inch, I received a confirmation of the importance of disturbing causes. Both boilers were reported to me to be precisely the same, but that one yielded from the same weight of coals in the same time about one-third less steam than the other. However, it turned out that there were two slight differences.

The duration of the experiments, the quantity of coals consumed, and the draught, were the same for each boiler.

No. 1. 7ft. diameter, 34ft. long, plates $\frac{8}{8}^{ths}$ of an inch thick.
No. 2. Do. do. do.
No. 1. Surface of fire-bars, 2ft. 8in. by 5ft. 6in. = 14.6 sq. ft.
No. 2. Do. 3ft. by 6ft. = 18.0 sq. ft.
No. 1. Evaporated 6.36 lbs. of water per pound of coal.
No. 2. Do. 8.00 lbs. do. do.

No. 1 was then opened and found to be dirty and slightly scaled, No. 2 being clean. On No. 1 being thoroughly cleaned and again experimented upon a higher result was obtained, thus:

No. 1 evaporated 7.39 lbs. of water per pound of coal.

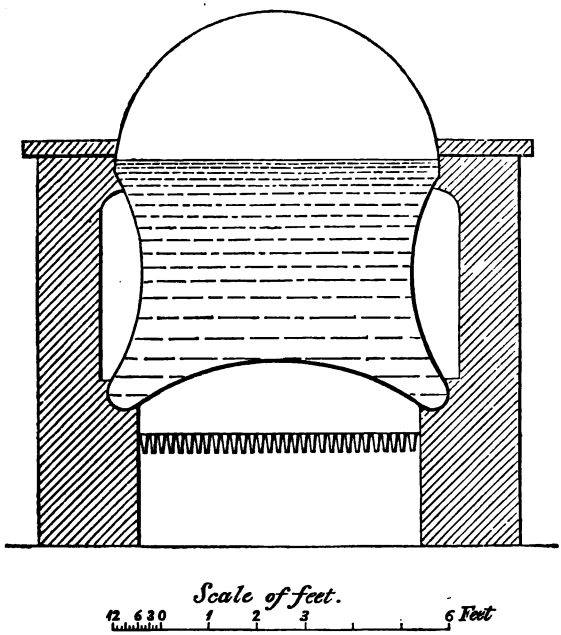
I could discover no other disturbing cause to account for the small yield from No. 1 beyond the comparative smallness of its fire-places, and accompaniment of dirt and slight scale. The evaporation is computed from water at the boil.

VII. Series of Experiments.

The boiler next experimented with was of the form as shown in the drawing usually called James Watt's "waggon-shaped boiler." It was 26 feet 6 inches long by 6 feet 6 inches in diameter.

Fire-place 5 feet 6 inches by 6 feet, equal to 33 square feet of fire surface.

Fig. 9.



Bars 16 inches from the top of the arch of the bottom of the boiler, half inch thick, with half inch spaces.

Plates $\frac{5}{16}$ths of an inch thick.

Depth of water over fire 4½ feet.

A flue carried round the boiler, with capital draught.

A corresponding series of experiments being made with this boiler gave a different result from that obtained with the last boiler; 8.80 lbs. of water from the temperature of 60° were evaporated by one pound of coal consumed; or, if we add one-sixth as equivalent to the heating of the feed water, the result will stand thus:

In the waggon-shaped boiler 10.26 lbs. of water were evaporated from 212° by one pound of coal.

In consequence of the facility with which dirt settles in the lower bends of this boiler it is more liable to be burned

at these parts than any of the boilers of other shapes I have experimented with. Indeed this circumstance forms an objection of some weight to the boiler. The only mode I have tried to obviate this difficulty, was to place two blow-off pipes, one in each bend, extending at least as far as the length of the fire, with numerous holes in their under sides so that the dirt could be sucked in and discharged from the boiler. I have observed, apart from the injurious effects of dirt or scale, that any substance touching the boiler where it is exposed to great heat, such as cast iron, bricks, mortar &c., will cause, sooner or later, a serious deterioration of the strength of the plates at these parts. If these points of contact are excited by the presence of moisture, the deterioration proceeds with the greatest rapidity.

VIII. Series of Experiments.

Fig. 10.

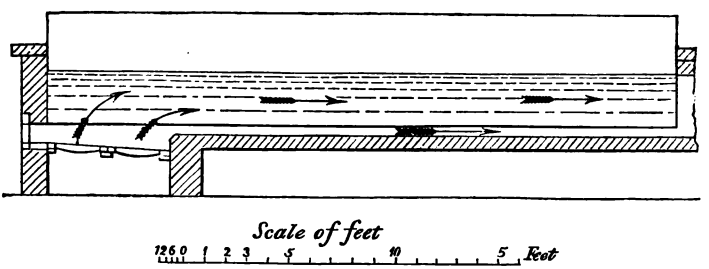
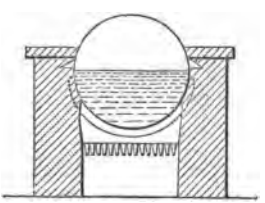


Fig. 11.



The third boiler experimented with was a plain cylinder, 33 feet long by 5 feet 6 inches in diameter.

Fire-place 4 feet 8 inches by 6 feet, equal to 28 square feet.

Fire-bars distant 12 inches at the front and 13 inches at the back from the bottom of the boiler.

Bars half inch thick with half inch spaces.

Boiler plate $\frac{5}{16}$ths of an inch thick.

Flame bed 5 inches from the boiler.

There was no flue, but the flame-bed was carried up nearly to the water-mark. The draught was thereby reduced to a thin and rapid current of 5 inches deep; under such circumstances the surface of the bottom of the boiler does not in any case coat with soot.

Quantity of water evaporated from a temperature of 60°.

	lbs.
When this boiler was worked at its ordinary speed, the average result from one pound of coal was, with a fire kept 5 inches thick	5.60
Ditto 7 inches thick	5.78
Ditto 10 inches thick	5.93
When worked briskly, and apparently pushed with a 7-inch fire.....	6.09
When in a dirty condition within, from a slight scale of sulphate of lime about the thickness of a sixpence or less, but in other respects under ordinary circumstances with a 7-inch fire.....	5.50
When made perfectly clean and keen internally by means of muriatic acid, with a 7-inch fire	6.45

This unusual activity produced by the muriatic acid continued only for a day or two. The boiler by that time, with the water used, became glazed or smoothed, and the keenness was destroyed.

This boiler at its ordinary speed evaporated, in 12 continuous working hours, 27,267 lbs. of water from the temperature of 60°, and when pushed 38,678 lbs.

The evaporating power of this boiler may therefore be

considered as 6.09 lbs. of water evaporated by one pound of coal.

Or by adding one-sixth to that quantity for the heating of the feed water from 60° to 212°, the evaporating power will be 7.1 lbs of water evaporated by one pound of coal.

I have already stated that as these experiments proceeded, the deviations from uniformity became less perplexing, and when such deviations occurred they were more readily accounted for. The following short table is given to illustrate the degree of accordance of such observations.

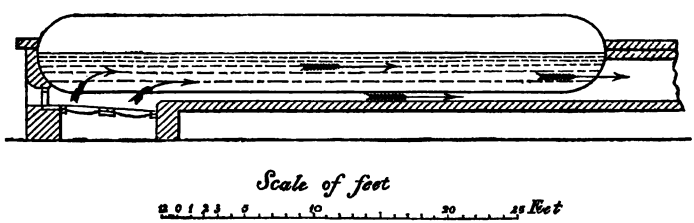
Table D.

Water evaporated from 60° per pound of coal burned.

By Thick Fires.		By Thin Fires.	
Lbs.	Observations.	Lbs.	Observations.
5.70	Monday.	5.63	
6.20		5.65	
5.97		5.74	
6.04		5.57	Saturday.
5.79		5.35	Monday.
5.90		5.57	
5.51	Monday.	5.36	Dirty coals.
5.93		5.64	
5.20	New fireman.		
6.00	Old fireman.		
&c. &c.			

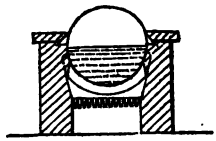
IX. Series of Experiments.

Fig. 12.



The next boiler experimented with was also plain and cylindrical, 42 feet long and 6 feet in diameter.

Fig. 13.



Fire-place 5 feet 1 inch by 6 feet 11 inch=35.1 square feet of fire surface.

Fire-bars placed 14 inches from the bottom of the boiler at the front, half inch thick with half inch spaces.

Boiler plate $\frac{5}{16}^{\text{ths}}$ of an inch thick.

Flame-bed, which was carried up to near the water mark, 6 inches from the bottom of boiler.

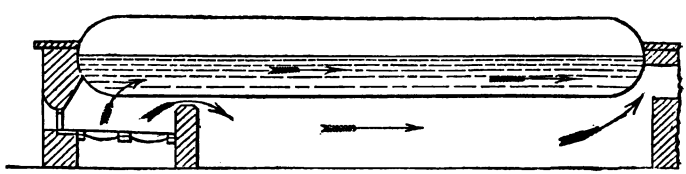
No return flue.

Water evaporated from 60° *by one pound of coal.*

	lbs.
When worked with the fire-bars 12 inches from the bottom of the boiler, the yield was	6.14
With an arrangement to cause the draught to impinge against the boiler bottom.....	6.30
By lowering the bars 2 inches, and under ordinary circumstances	6.46
By the introduction of a series of half-bridges, whereby the draught was dispersed and caused to impinge in all directions	6.69
By lowering the bars 11½ inches more (whole distance 25½ inches), under ordinary circumstances	5.11

This boiler had been fitted up previously to these experiments, and worked for some time as seen in *Fig.* 14.

Fig. 14.



It had been thus fitted to meet theoretical considerations in connection with smoke burning. When altered to the condition as seen in figures 12 and 13, a saving accrued of at least 30 per cent. on the consumption of coal, as practically estimated.

The evaporating power of the present boiler, which is of the same class as the last, but greater in diameter and longer, may be considered as 6.46 lbs. of water at 60° evaporated by one pound of coal.

Or adding one-sixth for the heating of the feed water from 60° to 212°, the power will be 7.54 lbs. of water evaporated by one pound of coal.

X. Series of Experiments.

Fig. 15.

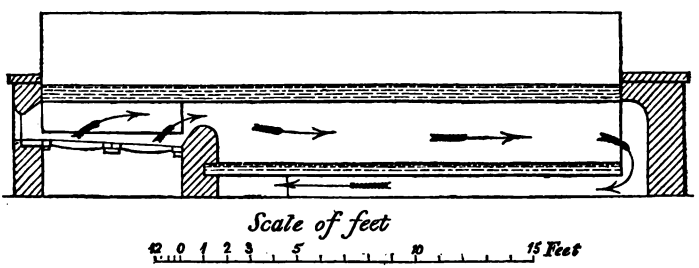


Fig. 16.

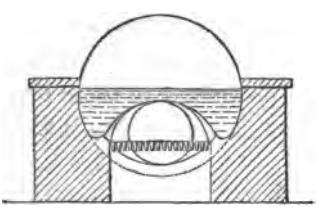
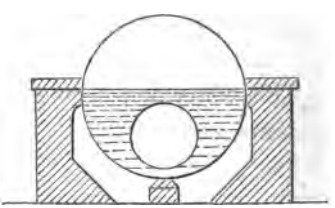


Fig. 17.



The next boiler was, as shown in the drawing, of the shape usually called the "Butterly," or, "Fishmouthed boiler." Its dimensions were as follows:

Length 25 feet by 7 feet in diameter.

Fire-place 4 feet 3 inches by 6 feet, equal to 25.4 square feet of fire surface.

Fire-bars 21 inches below the crown or arch forming the bottom of the boiler.

Bars half inch thick with half inch spaces.

Internal flue 29 inches in diameter.

Boiler plates $\frac{5}{16}^{ths}$ of an inch thick.

Flues were carried round the boiler.

When this boiler was worked at its usual speed and in ordinary circumstances the evaporating power was 8.33 lbs. of water evaporated from a temperature of 60°.

At its ordinary speed it evaporated 26,440 lbs. of water from the temperature of 60° in 12 successive working hours.

By adding to the above result one-sixth for the heating of the feed water from 60° to 212°, the evaporating power of the boiler will be 9.72 lbs. of water evaporated by one pound of coal.

This boiler became a general favourite, and was used with the smallest of the cylindrical boilers (*Fig.* 10) in most of my subsequent experiments.

In making these experiments I found it the more necessary as I proceeded to guard myself against all bias from reasoning of a purely speculative character, and to avoid the smallest approach to anticipating results. The theoretical part of the inquiry will be best treated separately, and I hope to take it up on some future occasion and to have the pleasure of laying the results before the Society.

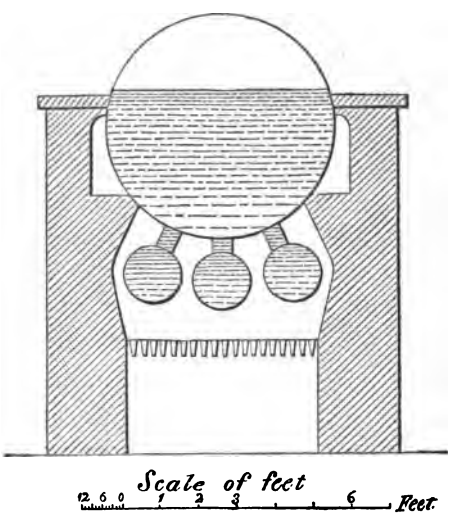
The simple object I have hitherto had in view was to learn how to consume as little coal and evaporate as much water as possible, and to acquire that knowledge by experiment alone. My method was to alter the flues, dampers, draughts, bridges, fire-bars, fires, coals, and stokers, and everything else connected with a particular boiler outside and inside, and then, if necessary, to alter back again,

until it appeared that I had obtained the most favourable condition for work, of the boiler under trial. To do all this required time and a considerable expenditure, as may be readily supposed. But the economical results which followed in greatly reduced consumption of coals, less frequent repair of boilers, and increased safety to the people employed about them, were always a sufficient motive to urge on the experiments. I may be allowed to say that the useful result in my case from these and other collateral experiments was a reduction of about 67 per cent. on the cost for coals per piece printed.

XI. Series of Experiments.

Attempts have been made to improve the boiler by enlarging that portion of its surface which is exposed to the direct radiation of the fire. A boiler possessing a contrivance for that purpose came into notice a few years ago, and was strongly recommended on theoretical grounds, as it presented a very large absorbing surface as compared

Fig. 18.

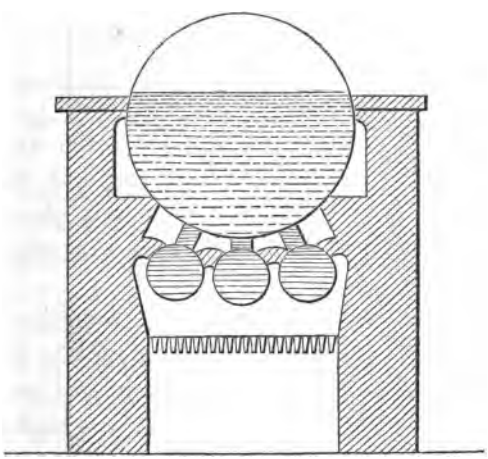


with the radiating fire surface. I was induced to have the cylindrical boiler, described under the VIII. head, altered to the new shape which is seen in *Fig.* 18.

Under the cylinder boiler of 5 feet 6 inches in diameter, three small cylinders, each of 1 foot 6 inches in diameter, extending the length of the main boiler, were placed, and filled with water from the upper boiler. A line carried across the fire-place measured 4 feet 8 inches, while a line carried in the same direction, but over the surface of the boiler bottom and its appendages, measured 17 feet 7 inches. When again used, the altered boiler proved unequal to the work of the original plain cylinder. It was found, indeed, when properly tested, to possess a remarkably low evaporating power, namely, from 5.5 to 5.8 lbs. of water at 212° per pound of coal.

The setting of the same boiler was then altered as seen in *Fig.* 19.

Fig. 19.

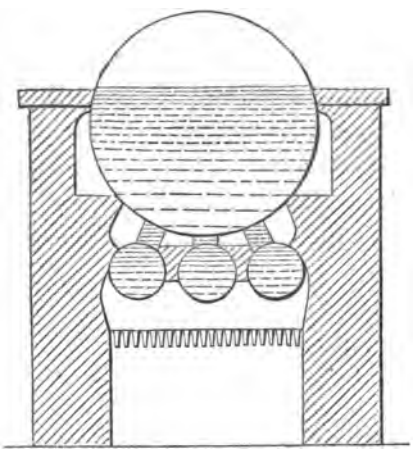


The lineal measurement of the absorbing surface was thus reduced to 10 feet 6 inches, while the line across the fire was the same as before. The new evaporating

power was not numerically determined, but an improvement in the power of the boiler was most evident. It was now possible to go on with it at least, and the boiler was supposed at the time to have recovered about 20 per cent. in power.

This result, however, being still unsatisfactory, I had the boiler altered as shewn in *Fig.* 20.

Fig. 20.

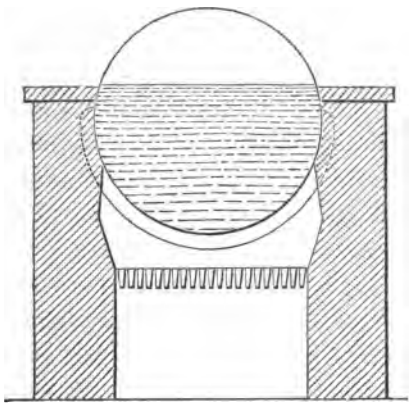


The direct absorbing surface of the boiler was now reduced to about 7 feet across, the fire remaining the same as before. Here again an improvement took place equal to at least an additional 8 or 10 per cent. as practically estimated.

The boiler was employed in this last condition for some time, doing the full work previously allotted to it as a plain cylindrical boiler. The uniform impression of those who attended it was, that the boiler gave a slightly better result in this condition than as a plain cylindrical boiler. There were, however, practical objections to the boiler in this form, the brickwork over the tubes was constantly shaking loose, and the tubes themselves became scaled and

dirty and could not easily be cleaned. When, therefore, the tubes were burned out, which took place in their third year, I decided to remove them, and replaced the boiler in its original simple form as seen below, which it will be remembered gave the result of 7.1 pounds of water evaporated.

Fig. 21.



Conclusions.

The more important conclusions to which I have been led from the experiments above described, may be stated as follows:

	Of Water evaporated from 212°, by 1 lb. of coal.
1. The boiler with the two internal fire-places, commonly called the "breeches boiler," worked as herein described, will yield	6.88 lbs
2. The "waggon-shaped boiler," worked as herein described, will yield	10.26 lbs
3. The smallest sized "plain cylindrical boiler," worked as herein described, will yield	7.1 lbs
4. The large sized "plain cylindrical boiler," worked as herein described, will yield.....	7.54 lbs
5. The "Butterly boiler," worked as herein described, will yield, for one pound of coals consumed	9.72 lbs

6. A supplementary boiler, under the circumstances above described, gives a saving of ..... 15 p c^t

7. When numerous cast iron pipes are substituted for the supplementary boiler, and mechanical means are adopted to keep their surface clear of soot, a saving may, under the circumstances described, be expected of ... .27.4 p c^t

8. The flues round a boiler, when they come to be coated to the extent of one-eighth of an inch with soot, are of little or no use in raising steam.

9. If the sides and bottom of the boiler exposed in the flues (where such flues exist) be scraped once a week clean from soot, a saving will ensue of about .... 2 p c^t

10. It is advantageous to convert the side flues at once into a wide flame-bed (where it is practicable), the flame and draught being spread out thin and going up nearly to the water mark on each side, and led straight on to the chimney. By such means the bottom of the boiler is preserved for its whole length free from soot, and there is an effective radiation from the hot bricks of the bed.

11. A very slight difference in the setting alone of the same boiler may readily produce a difference in the result amounting to ..... 21 p c^t

In extreme cases of bad setting, as may be seen at *Fig.* 14, of course a much larger loss than this may be looked for.

12. The difference between a boiler of one shape properly set, and another boiler of a different shape improperly set, but both clean and in good order, may readily amount to ..... 42 p c^t

In some cases even a greater difference is possible.

13. A difference in the mode of firing only may produce a difference in the result of ..... 13 p c^t

I refer here to a difference such as will be produced by withdrawing the regular and practised stoker, and substituting for him an equally intelligent and careful man, but who has not been accustomed to the boiler or the coal employed. A man who is careless or dishonest will occasion a loss, the extent of which can scarcely be calculated. All injurious influences upon firing are greatly aggravated in very large fire-places (say of six feet by seven feet) from the physical inability of the stoker to keep the bars equally covered.

14. The loss arising from the scale of sulphate of lime on the inner surface of the boiler, of not more than one-sixteenth of an inch, amounted to 14.7 p c^t

15. Neither wet coals, nor coals which had been out of the pit for three years, nor wet weather, nor a variation of temperature of the atmosphere from 40° to 70°, produced any appreciable difference of result.

16. Windy weather invariably gave a high result.

17. A moderately thick and hot fire, with a rapid draught, uniformly gave the best result.

The coals used varying from the condition of dust to that of pieces four inches in diameter, it was found advantageous to keep the fire at a general thickness of from six to seven inches, in order to insure the bars being covered at all times; and as it is found, that in all boiler fires the coals are consumed more quickly

at some parts of the bed than at others, such parts were supplied with extra quantities. This irregularity in the burning of boiler fires is one of the troublesome "disturbing causes" in boiler experiments, the more so that such quick burning localities shift slightly with variations of the wind and other changes. The fire-bars were always kept open, so that the ashpit was always bright and clear. The fire itself was "clinkered" at least once, and often twice in a day.

18. This law, as regards rapid combustion, was reversed on the small scale. In the usual evaporations commonly taking place in Print Works, where pans and open boilers are employed, varying in capacity from ten to three hundred gallons, the rate of evaporation per pound of coal is very low, varying from two to four and a half pounds of water evaporated from 212° for one pound of coal; and the law of evaporation from such vessels is about as follows: If the time is reduced to one half, the evaporation per pound of coal will be reduced also, to the extent of from 10 to 20 p c^{t}.

19. The difference in the results obtained by a change in the coal used amounted to .... 11 p c^{t}

20. The same coal reputed to be from the same pit varied in its evaporating power to the extent of 6 p c^{t}

21. The higher the water stood in the boiler within certain unknown limits, the better was the result. In some experiments this advantage was in the proportion of 1 per cent. for every six inches of depth of water.

22. As regards the effect of pressure on evaporation, the general result of these experiments is in favour of the sup-

position that the rate of evaporation of water per pound of coals increases with and bears some ratio to the pressure under which the steam is generated. In connection with this idea I may mention a singular fact which may be supposed to give it support, or may, perhaps, find some other practical explanation. If a boiler be used exclusively for the purpose of heating water and liquors in a dyehouse, and be capable of working that dyehouse just sufficiently when the steam is at 10 lbs. pressure, it will be found unable to do so when the pressure falls to 7 lbs., and still less able to do its work when the pressure further falls to 2½ lbs. By repeated experiments on the large scale I find that the loss incurred by working steam at 2½ lbs. as compared with steam at 10 lbs., *an equal quantity of coals always being burned in the same time,* is as follows: —

Available power of the boiler to heat dye cisterns:

With the steam at 10 lbs. pressure=100
„ „ 7 lbs. pressure= 92
„ „ 2½ lbs. pressure= 77

I have not experimented at higher or lower pressures than those stated.

23. While we may reasonably look for advantage from improvements in the construction of the fireplace, in the management of the fire itself, in the construction of the flame-bed and in adaptation of draught, in the form of the boiler, in the addition of separate supplementary heating surface, and in cleanliness, and by all these means effect a great saving in the consumption of coals, we cannot at the same time expect much saving from extension of flue space when allowed to coat with soot, nor from a greater length of boiler than four times the length of the fire-place.

24. With a view to the prevention of the "scaling" of boilers, or the coating with sulphate and carbonate of lime and mud, I have experimented with the following substances: caustic soda, quicklime, muriatic acid, soap liquor,

sawdust, spent madder, and logwood chips, with various degrees of advantage. Muriatic acid proved a most effective agent, indeed somewhat too much so. Although I never added more than two-thirds of the quantity of that acid necessary to saturate the lime known to be in the water as carbonate, yet the action continued not only to the point of the disappearance of the remaining third of the carbonate of lime, but went further. It had an action, also, on the oxide of iron surrounding the rivets, and existing between the edges of the plates; so that after six weeks, the usual duration of an experiment, it was found that the iron plates were not only clean and bright like silver, but also that the joints had been acted on to a considerable extent. The rust did not appear to be dissolved, but only loosened, nor did the metal of the plates appear to be corroded, for water from the boiler tested daily showed no iron.

The water used in these experiments was from the Lancashire "Red Rock," hard in quality, containing both sulphate and carbonate of lime, with a little iron. It required forty measures of Dr. Clarke's soap test liquor to soften it. The scale, when allowed to form, was composed in 100 parts of 22.5 carbonate of lime, and 77.5 sulphate of lime.

Two facts were noticed as regards the tendency of hard water to scale. First, that the *sulphate of lime* separates from the water only when in contact with solid objects, such as the bottom and sides of the boiler, or solid matters floating in the boiler, such as sawdust and logwood chips (hence their use); and that no precipitation takes place until the water has been concentrated by continued evaporation down to the condition of a saturated solution of sulphate of lime, or to that point which may be termed the "salting point." Secondly, *carbonate of lime* and mud are principally deposited in the midst of the water, and have but little

disposition to adhere to the boiler, unless cemented by the sulphate of lime.

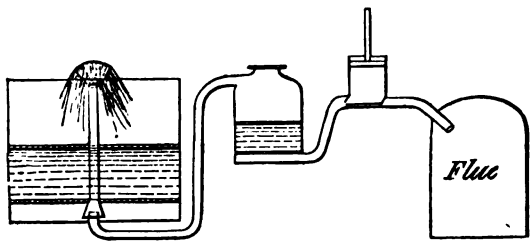
It is found accordingly, that no scale of much consequence will be formed on engine boilers, with even such hard water as is used, if 100 gallons of water daily, and 200 additional gallons on Saturdays be run off through the usual mud machine, and if the boiler on every sixth Saturday be further entirely emptied and swept out. There is little loss incurred by proceeding in this manner, as the chief discharge may take place at the close of each day, while a large profit accrues by the saving of coals and the increased durability of the boiler, with security from danger to all concerned with it.

25. In the five boilers most used in these experiments, the highest results were obtained when all the air required to consume the coals was introduced into the fire-place through the fire-bars. The introduction of additional air elsewhere uniformly occasioned loss, whether it was admitted by means of the furnace door during two minutes after firing, or for a longer or shorter period, or through a regulating slit above the furnace door, or through apertures of larger or less size and more or less numerous, in the sides of the furnace, in the front of the bridge, or on the top or back of the bridge.

26. Perfect combustion of black smoke was never arrived at, without the introduction of additional air. The consequent loss was a little over 1 per cent.

This loss, by the addition of air over the fire, will be uuderstood when we consider the smallness of the weight of black smoke. The following experiments may convey some information on this point. I had in view the possibility of working bleaching keirs with hot air from the flue of the furnaces instead of steam. The apparatus used is shown in the drawing: —

Fig. 22.



I drew the hot air and smoke out of the main flue by means of a pump at a temperature of perhaps 600°, forced it through the intermediate washing vessel, in which was kept a small supply of water, and where mechanical means were adopted to subdivide the stream of air, and passed it on to the keir. The result of these experiments was a failure as regards the heating of the keir. I passed the hot air at a high velocity, namely, about 130 cubic feet per minute, and found that the heat communicated by the mass of air was barely sufficient to sustain the contents of the keir at a temperature not above 190° F., and that it took from six to eight hours to bring it up to that point. Incidentally, however, I had the means of estimating the weight of black smoke. The hot air and smoke used were from furnaces that produced dense black smoke after each fresh supply of coals. The washing apparatus intercepted the soot in such a perfect manner, that the water in the keir was not more than discoloured at the termination of the experiment. The black residuum in the washer, composed of dust and carbon, was carefully examined, but not analysed. I formed an opinion to the best of my judgment as to the amount of carbon, and as to its weight relative to the coal consumed in heating the air and smoke drawn by the pump, which was that the weight of black smoke does *not exceed* the one thousandth part of the coals consumed. The power of black smoke as

it issues from the chimney-top to discolour the atmosphere is prodigious, but its weight is inconsiderable.

This discolouring power is rapidly abated and soon destroyed by the aggregation of the particles through electrical or other causes, and quick precipitation ensues, each particle being assisted in its descent by an associated burden of dust and other matters. The dense heavy cloud we see floating off to leeward of such large cities as Manchester, is not composed of black smoke; indeed, it may be questioned if, at the distance of half-a-mile from its source, and unless drifted like sand before the gale, there is then a single particle of black smoke in the atmosphere. Those who have "crofted pieces" under the current of their chimney can estimate the correctness of this remark as to the rapid precipitation of black smoke. The dust which descends, associated with the black smoke, has been invariably found to contain iron. This may be shown on the most minute scale by placing a particle of the soot between the wet folds of a pink piece dyed with madder at that point of the process when the piece has been washed out of the "tin." The chemical law of displacement in this experiment is beautifully shown by the iron taking the place of the alumina that formed the basis of the pink colour, and itself producing a black with the liberated alizarine. What we see floating in the atmosphere is dust associated with the volatile constituents of the coal which have been distilled off through imperfect combustion. It is scarcely in place in this paper to go further into this important subject; but I may remark that this rapid disappearance of black smoke by precipitation, more particularly in dry weather, is one of the marked tests of the proper action of the fires below, while if the smoke hangs lazily, keeps together, and is visibly black for a greater distance, we may be certain that something is going up

the chimney as well as black smoke, and in serious quantities, which ought to have been burned in the fire itself. The presence of such volatilized oils appears to have the effect of sustaining the black smoke by hindering its aggregation and precipitation.

III. — *On the triedral partitions of the* X*-ace, and the triangular partitions of the* X*-gon. By the* Rev. Tho. P. Kirkman, A.M., F.R.S., *Rector of Croft with Southworth, and Honorary Member of the Literary and Philosophical Society of Manchester.*

Read November 17th, 1857.

1. Perhaps the simplest proof that can be given of the relation,

$$F+S=E+2,$$

among the F faces, S summits, and E edges of a polyedron, is the following. Let *e* be any edge joining the summits *a b* and the faces A B, and let *e* vanish by the approach of *b* to *a*. If A and B are neither of them triangles, they both remain, though reduced in rank and no longer collateral, and the figure has lost one edge *e* and one summit *b*. If B is a triangle and A no triangle, B vanishes with *e* into an edge through *a*, but A remains. The figure has lost two edges of B, one face B, and one summit *b*. If B and A are both triangles, B and A both vanish with *e*, five edges forming those triangles are reduced to two through *a*, and the figure has lost three edges, two faces, and the summit *b*. In any of these cases, whether one edge and one summit vanish, or two edges disappear with a face and a summit, or three edges with a summit and two faces, the truth or falsehood of the equation

$$F+S=E+2$$

remains unaltered. By causing all the edges which do not meet any face to vanish, we reduce the figure to a pyramid upon that face. Now the relation is true of the pyramid; therefore it is true of the undiminished polyedron.

THEOREM A. *In every* $(x+1)$*-edron P having an* x*-gona, face X, there are at least two triangular faces.*

For there is no face of the $(x+1)$-edron not collateral with X: wherefore the edges of P which do not meet X can enclose no space, but must form a broken line whose terminations are $p_1\ p_2 \ldots$, two or more. From each point $p_1\ p_2 \ldots$ two or more edges must pass to meet the base, since all the summits are at least 3-edral. That is $p_1\ p_2 \ldots$ are vertices each of one or more triangles collateral with X. *Q. E. D.*

2. The partitions of an x-ace, (x rays diverging from a point,) correspond to those of the x-gon. The diagonal joining the m^{th} and n^{th} summits of the latter answers to the intersection of the m^{th} and n^{th} faces about the former. The number of triedral partitions of the x-ace is that of the triangular partitions of the x-gon, in which of course no two diagonals cross each other. A triangular partition of a non-planar or skew x-gon, crowned by a simple x-ace whose rays pass through its x summits, is an $(x+1)$-acron with only triangular faces. A triedral partition of an x-ace, standing on an unpartitioned x-gon, is an $(x+1)$-edron whose summits are all triaces.

If we can enumerate the $(x+1)$-edra on an x-gonal base which have all their summits triaces, we shall have the solution of the problem of the triangular partitions of the x-gon, or of the triedral partitions of the x-ace. The object of this memoir is to show that this problem is very conveniently attacked by the consideration of these $(x+1)$-edra, and that this mode of solution is, where all modes seem difficult, as easy as any. And the view here taken of the question is entirely new.

3. Let P be an $(x+1)$-edron in an x-gonal base, having only triedral summits. P has f triangular faces, $f \not< 2$. If $f = \frac{1}{2}x$, no edge meets the base which is not the side of a triangle. If $f < \frac{1}{2}x$, there are $x-2f$ edges (j) meeting the base which are not sides of triangles. And f cannot be $> \frac{1}{2}x$, for if it were, there would be two triangles about the base having a common vertex out of it, which would be no triace, unless the figure were a tetraedron.

Each of these $x-2f$ edges (j) connects a summit of the base with one (i) out of it. If we suppose that edge j to evanesce by the coincidence of the two faces about it, a summit disappears of the base, and a summit (i) disappears among those out of the base. Let all the $x-2f$ edges then disappear; the base is now $2f$-gonal, and has the same f triangles collateral with it, the greatest number possible, as has just been shown. And all the $x-2f$ summits i out of the base that have vanished have disappeared from edges k that do not meet the base.

4. It follows that all $(x+1)$-edra having f $(< \frac{1}{2}x)$ triangles, can be constructed from the $(2f+1)$-edra having f triangles without touching those triangles, by joining points (i) in edges not meeting the base and not in triangles, to point in base edges, the face in which any such joining line (j) is drawn being supposed fractured into two faces thereby, and every such operation adding a face, two summits, and three edges, but altering and adding no triangles. Every $(2f+1)$-edron having f triangles is evidently made by cutting away every base summit of a certain $(f+1)$-edron on an f-gonal base, which may have any number of triangles from 2 to $\frac{1}{2}f$, or $\frac{1}{2}(f-1)$, as f is even or odd.

If the number of these $(f+1)$-edra is known, the number of $(2f+1)$-edra having f triangles is also known; and if the whole number of ways be known in which m joining lines (j) above described can be drawn upon these

$(2f+1)$-edra, the number of $(2f+m+1)$-edra on $(2f+m)$-gonal base having f triangles is determined. And thus the entire number of $(x+1)$-edra on an x-gonal base can be found, and classified according to the number f of their triangular faces.

It is to be understood that in this enumeration no $(x+1)$-edron shall be either the repetition in a different position about the base, or the reflected image of any other. The avoiding of these repetitions is the great difficulty of our problem.

5. Polyedra, whatever be the rank of their faces and summits, as well as partitions either of the polygon or the polyace, fall into two leading classes, the *irreversible* and the *reversible*.

An irreversible is in no position identical with its reflected image. A reversible is in certain positions identical with its reflection.

Again, an irreversible regarded at one point of its base may be identical with itself regarded at another or at two other points of the base. If at one other only, it is called a doubly irreversible; if at two others, it is called a triply irreversible. In the former, the sequence of faces about the base repeats itself once in the circuit; in the latter, it repeats itself twice: that is, it is three times read in the circuit. A singly irreversible is in no two positions identical with itself.

In like manner, a reversible may be singly, doubly, or triply reversible, exhibiting a configuration about the base that occupies the whole circuit, half the circuit, or one third of it.

A singly reversible can be placed in two positions before a mirror so as to be identical with its image, in which position a bisecting axial plane is perpendicular to the mirror. A doubly reversible has four such positions in two axial planes at right angles to each other. A triply

reversible has six such positions in three axial planes making equal angles with each other. All this requires the supposition, which we are at liberty to make, that the base of the polyedron is a polygon regularly inscribed in a circle, on which the whole figure is constructed as symmetrically as possible.

The three following are irreversibles, single, double and triple, standing on 7-gonal, 6-gonal and 9-gonal bases:

5436353, 435435, 643643643,

of which the third *e.g.* shows a hexagon, a 4-lateral and a triangle thrice read in that order about the base.

The three following are reversibles, single, double and triple, on 5-gonal, 8-gonal and 6-gonal bases:

53443, 53635363, 535353.

6. A sequence four times read can never appear if the summits are all triedral. For in such a sequence the triangles may be supposed to become infinitely small, and in the result of their evanescence the triangles again might be supposed to disappear, and so on till a figure appeared incapable of further reduction by the evanescence of triangles. But this figure would still have a four-fold sequence, the original periods of configuration having been treated alike: that is, the figure would be a pyramid in a quadrilateral base, having a 4-ace, contrary to hypothesis.

The effect of the evanescence of a triangle is to reduce by one side the rank of the face on either hand; thus, by the vanishing of the triangles,

643643643 becomes 535353,

and 53635363 becomes 3434.

7. PROBLEM *a*.

Let P be singly irreversible $(2x+1)$-edron, having a $2x$-gonal base and x triangular faces. Required: the number of $(2x+m+1)$-edra that can be constructed from it by the process of Art. 4, to have also x triangles.

As all the S summits are triedral, or have

$$2E=3S, \quad \text{also}$$
$$F+S=E+2, \quad \text{whence}$$
$$3F+2E=3E+6,$$
$$E=3F-6=6x-3,$$

of which $6x-3$ edges $2x$ are in the base, and $2x$ are sides of triangles, leaving $2x-3$ edges not meeting the base and none of them in a triangle.

8. If we draw in any non-triangular face h collateral with the base a line j, from a point of the base to a point i of any edge of h which does not meet the base, and consider h fractured along that line into two faces, the resulting $(2x+2)$-edron (Art. 4) is still singly irreversible; for the new summit of the base is the only one not in a triangle. If we draw another such line j either in h or some other non-triangular face, the result is still singly irreversible; for, by definition (Art 5), the new summit of the base is part of a configuration which nowhere else is read about the base: and in like manner, any number of such operations upon a singly irreversible subject will give a result singly irreversible, and no two of these results can be alike, because in our singly irreversible subject of operation the configuration read round the base from any face is different from that read round from any other.

Our problem is to determine in how many different ways such operations can be performed on our subject P.

There will be added m summits i out of the base, which will be planted on the $2x-3$ edges k not in triangles. The number of ways in which these m points can be distributed on those edges, w on any edge, $(w \leqq o)$ is, putting $a^{b|1}=a\cdot\overline{a+1}\cdot\overline{a+2}\ldots(b \text{ factors})$,

$$\frac{(m+1)^{2x-4|1}}{1^{(2x-4)|1}},$$

And as the joining line j through any of these points can be drawn to the base through either of the faces meeting

in the edge k containing the point i, the whole number of arrangements of the m new lines j is

$$\frac{(m+1)^{2x-4\,|\,1}}{1^{(2x-4\,|\,1}} \cdot 2^m.$$

If then $\mathrm{I}(2x+1,x)$ be the number of singly irreversible $(2x+1)$-edra having x triangles, and

$$\mathrm{II}(2x+m+1,x)$$

denote the number thus constructible of $(2x+m+1)$-edra singly irreversible that have x triangles, we have proved that

$$\mathrm{II}(2x+m+1,x) = \mathrm{I}(2x+1,x) \cdot 2^m \frac{(m+1)^{(2x-4\,|\,1}}{1^{(2x-4\,|\,1}}.$$

For any one of the $\mathrm{I}(2x+1,x)$ being taken for the subject of operation, will give the same number of results as our subject P.

The number $\mathrm{II}(2x+m+1,x)$ is only part of $\mathrm{I}(2x+m+1,x)$, the whole of the singly irreversible $(2x+m+1)$-edra that have x triangles, namely that part which reduces, by the process of Art. 3, to the irreversibles $\mathrm{I}(2x+1,x)$. The remainder of the number $\mathrm{I}(2x+m+1,x)$ we shall find in the proper place.

9. Problem *b*. To determine the number of doubly irreversible $(4x+m+1)$-edra having $2x$ triangles reducible (Art. 3) to doubly irreversible $(4x+1)$-edra having $2x$ triangles.

It is to be understood here and everywhere in this memoir, that we are considering only $(x+1)$-edra which have an x-gonal base, and all their summits triedral.

A doubly irreversible must have an even number $2x$ of triangles, because any sequence of faces about the base is twice read in the circuit; that is, the triangles and the configurations about them are corresponding pairs.

Let P be the doubly irreversible $(4x+1)$-gon, the subject of our operations. That our results be also doubly

irreversible, it is necessary that m be even, for the lines j must form corresponding pairs.

We have (Art. 6), $4x-3$ edges k not meeting the base, on which to distribute our m points i, an odd number of edges. Now, our subject is such that if any plane perpendicular to the base is drawn to bisect it, and to contain no edge k, the configuration on one side of that plane is a repetition of that on the other. There is therefore one edge k' bisected by that plane, and for every point i planted on one half of k' we must plant another i on the other half; and every operation on one side that plane must be repeated in the proper place on the other side, if our result is to be doubly irreversible.

We have, then, to consider in how many ways $\frac{1}{2}m$ points i can be laid on $\frac{1}{2}(4x-3+1)=2x-1$ edges k. The number of operations by the preceding article is

$$\frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m}{2}},$$

which are to be repeated in direct order on both sides the bisecting plane.

These results are all different, because no sequence can be read round the base to begin at two different points on one side of our bisecting plane. And no more results could be obtained by varying the position of that plane, since every possible disposition of $\frac{1}{2}m$ lines j has already been made on half the figure.

If the $\mathrm{I}^2(4x+1,2x)$ be the entire number of doubly irreversible $(4x+1)$-edra having $2x$ triangles, and $\mathrm{I}^2\mathrm{I}^2(4x+m+1,2x)$ be the number of doubly irreversibles thus constructible, we have

$$\mathrm{I}^2\mathrm{I}^2(4x+m+1,2x)=\mathrm{I}^2(3x+1,2x)\cdot 2^{\frac{m}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}},$$

for the portion thus obtainable of the whole number $\mathrm{I}^2(4x+m+1,2x)$ of doubly irreversible $(4x+m+1)$-edra

having $2x$ triangles. The remaining portion we shall find presently.

10. Observe that the number $\mathrm{I}^2\mathrm{I}^2$ above written is always to be considered zero, if $\frac{1}{2}m$ is not integer; and in all future expressions of this kind, the occurrence of $\frac{m}{a}$ or $\frac{m-e}{a}$ not an integer in a factorial is to be considered equivalent to that of the factor zero before the factorial. We may thus dispense with the use of circulating constants.

PROBLEM *c*. To determine the singly irreversible $(4x+m+1)$-edra having $2x$ triangles reducible (Art. 3) to doubly irreversible $(4x+1)$-edra having $2x$ triangles.

If we distribute m points i in every possible way on the $4x-3$ edges j of the subject P operated on in the preceding article, and draw m lines j in every possible way, we shall obtain

$$\frac{(m+1)^{4x-4\,|\,1}}{1^{4x-4\,|\,1}} \cdot 2^m$$

results. Among these will evidently be found all those of the preceding article, once, and once only. The remaining results will be simply irreversible, and each will be twice constructed; for the result P′ presenting a single sequence of configuration read round the base from a face h, will be identical with another result P″, having the same reading round the base beginning at the face of P opposite to h.

If then from our total number of results we subtract those of the preceding article, the remainder divided by two will be the correct number of single irreversibles required.

That is, if $\mathrm{II}^2(4x+m+1,2x)$ be this correct number,

$$\mathrm{II}^2(4x+m+1,2x) = \tfrac{1}{2}\left\{\frac{(m+1)^{4-4\,|\,1}}{1^{4x-4\,|\,1}} \cdot 2^m - \frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}} \cdot 2^{\frac{m}{2}}\right\} \cdot \mathrm{I}^2(4x+1,2x),$$

a portion of the whole number $I(4x+m+1,2k)$, of $(4x+m+1)$-edra singly irreversible having $2x$ triangles.

11. Problem *d*. To determine the number of triply irreversible $(6x+m+1)$-edra having $3x$ triangles, which are reducible (Art. 3) to triply irreversible $(6x+1)$-edra having $3x$ triangles.

We must operate on the third part only of the $6x-3$ edges k (Art. 6), and draw $\frac{1}{3}m$ lines j. This gives us

$$\frac{\left(\frac{m}{3}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m}{3}}$$

results. If we repeat these operations in direct order on the other two thirds of the subject, we shall, as is evident from the reasoning of Art. 9, have all possible configurations and none twice repeated among them. The number of triangles must of course be $3x$.

If then $I^3I^3(6x+m1,3x)$ be the triply irreversibles, thus constructible from the whole number $I^3(6x+1,3x)$ of triply irreversibles with $3x$ triangles, then is

$$I^3I^3(6x+m+1,3x)=I^3(6x+1,3x)\cdot\frac{\left(\frac{m}{3}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m}{3}},$$

a portion of $I^3(6x+m+1,3x)$, the triply irreversible $(6x+m+1)$-edra having $3x$ triangles. Others will hereafter be found.

No doubly irreversible can be constructed on a triply irreversible subject, for a plane bisecting the base does not here show the configuration on one side of it repeated exactly on the other. We proceed then to

12. Problem *e*. To determine the number of singly irreversible $(6x+m+1,)$-edra having $3x$ triangles reducible to) $6x+1)$-edra triply irreversible having the same number.

If we plant m points i on our $6x-3$ edges k in every possible way, and draw the lines j, we obtain

$$\frac{(m+1)^{\overline{6x-4\,|\,1}}}{1^{\overline{6x-4\,|\,1}}}\cdot 2^m$$

results. Among these every result in the preceding problem will occur once, and once only, and every singly irreversible will occur three times, because the same sequence will have been begun and carried round from three similar starting points, say three h-gonal faces, of our subject. We have then

$$\mathrm{II}^3(6x+m+1,3x)=\mathrm{I}^3(6x+1,3x)\cdot\tfrac{1}{3}\left(\frac{(m+)1^{\overline{6x-4\,|\,1}}}{1^{\overline{6x+4\,|\,1}}}\cdot 2^m - \frac{\left(\frac{m}{3}+1\right)^{\overline{2x-2\,|\,1}}}{1^{\overline{2x-2\,|\,1}}}\cdot 2^{\frac{m}{3}}\right)$$

as a portion of the number $\mathrm{I}(6x+m+1,3x)$.

13. Problem *f*. To determine the number of singly reversible $(2x+m+1)$-edra reducible to singly reversible $(2x+1)$-edra with x triangles (Art. 3).

In a singly reversible one vertical plane and one only can be drawn bisecting the base, such that the configuration on one side of it is a repetition in reversed order of that on the other. The number of edges k is $2x-3$, an odd number. This bisecting plane, which we may call the axial plane, or plane of reversion, will therefore either contain one edge k, or bisect one.

First let the axial plane contain an edge k': this edge can receive no point i; for a line j being drawn from it on one side the plane would require, in the reversal of operations on this other side, another line j to be drawn from the same point on the other side, which would make i a 4-ace. We have then $\frac{1}{2}m$ points i to plant on $\frac{1}{2}(2x-3-1)=x-2$ edges k on one side of the plane. This gives us

$$\frac{\left(\frac{m}{2}+1\right)^{\overline{x-3\,|\,1}}}{1^{\overline{x-3\,|\,1}}}\cdot 2^{\frac{m}{2}}$$

results, so that m is of necessity even.

Let $\underset{kax=1}{R}(2x+1,x)$ denote the whole number of singly reversible $(2x+1)$-edra having x triangles, in which one edge k' is axial or in the axial plane, (the subindex $kax=1$ signifying one is axial,) among which number is the subject of our operations. Then we have

$$\underset{kax=1}{RR}(2x+m+1,x)=\underset{kax=1}{R}(2x+1,x)\cdot\frac{\left(\frac{m}{2}+1\right)^{x-3\,|\,1}}{1^{x-3\,|\,1}}\cdot 2^{\frac{m}{2}},$$

for the number of our results, which form a portion of $\underset{kax=1}{R}(2x+m+1,x)$, the $(2x+m+1)$-edra singly reversible having x triangles, in which the axial plane meets none, *i.e.* contains one, of the edges not meeting the base.

Next let the axial plane bisect an edge k'.

We can now operate upon k' on both sides the bisecting plane, without producing 4-aces. It is easily seen that our results are, if m is even,

$$\frac{\left(\frac{m}{2}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\cdot 2^{\frac{m}{2}},$$

and we obtain the equation

$$\underset{kax=0}{RR}(2x+m+1,x)=\underset{kax=0}{R}(2x+1,x)+2^{\frac{m}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}},$$

as a portion of $\underset{ax=0.}{R}(2x+m+1,x)$, the whole number of $(2x+m+1)$-edra singly reversible having x triangles, in which no edge k is axial, as denoted by the subindex $kax=0$. The base in both these sub-classes is even angled $(2x+m)$-gonal.

If however m is odd, we can produce as just pointed out, with the even number $m-1$ operations,

$$\frac{\frac{m-1}{2}+1^{x-2\,|\,1}}{1^{x-2\,|\,1}}\cdot 2^{\frac{m-1}{2}}$$

results, and then plant one point more in the axial plane at the centre of the edge k', from which of course two lines j can be drawn, doubling this number.

The $(2x+m+1)$-edra thus produced have no axial edge k', but they have all an axial point i and line j, which neither of the previous sub-classes of this article have. We may write therefore,

$$\mathrm{RR}(2x+m+1,x) = \mathrm{R}(2x+1,x)_{jax=1} \cdot \frac{\left(\frac{m-1}{2}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}} \cdot 2^{\frac{m+1}{2}},$$

the subindex $jax=1$ denoting that these have all a summit i in the axial plane. This then is a portion of the number $\mathrm{R}(2x+m+1,x)_{jax=1}$ of reversible $(2x+m+1)$-edra having x triangles, and having the base odd-angled, $(2x+m)$-gonal, as must necessarily be the case when the axis of reversion passes through one summit of the base.

14. Although no reversible can be obtained from an irreversible subject of operation, the converse is not true; and we proceed next to

PROBLEM g. To determine the number of irreversible $(2x+m+1)$-edra reducible to singly reversible $(2x+1)$-edra having x triangles.

These will all be *singly* irreversible, because the subject of operation stands alike in no two positions, read in the same direction. We have $2x-3$ edges k, on which to place m points, giving us

$$\frac{(m+1)^{2x-4\,|\,1}}{1^{2x-4\,|\,1}} \cdot 2^m$$

results, among which all those of the preceding problem will occur once and once only, while each of the remaining irreversibles will occur twice, in one result as the reflected image of another result, because of the reversible character of the subject. We obtain, therefore, dividing this remainder by 2,

$$\mathrm{I\,R}(2x+m+1,x) = \tfrac{1}{2}\mathrm{R}(2x+1,x) \cdot \frac{(m+1)^{2x-4\,|\,1}}{1^{2x-4\,|\,1}} \cdot 2^m$$

$$- \tfrac{1}{2}\mathrm{R}(2x+1,x)_{kax=1} \cdot 2^{\frac{m}{2}} \cdot \frac{\left(\frac{m}{2}+1\right)^{x-3\,|\,1}}{1^{x-3\,|\,1}} - \tfrac{1}{2}\mathrm{R}(2x+1,x)_{kax=0}$$

$$\times \left\{ 2^{\frac{m}{2}} \cdot \frac{\left(\frac{m}{2}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}} + 2^{\frac{m+1}{2}} \cdot \frac{\frac{m+1}{2}^{x-2\,|\,1}}{1^{x-2\,|\,1}} \right\}$$

which is a portion of the number $\mathrm{I}(2x+m+1,x)$ of singly irreversible $2x+m+1$)-edra having x triangles.

$\mathrm{R}(2x+1,x)$ here stands for the whole number of singly reversible $(2x+1)$-edra having x triangles, *i.e.*

$$\mathrm{R}(2x+1,x) = \mathrm{R}(2x+1,x)_{kax=1} + \mathrm{R}(2x+1,x)_{kax=0}.$$

15. Problem *h*. To determine the number of doubly reversible $(4x+m+1)$-edra having $2x$ triangles.

No mention is made here of the subject to which these are reducible, because it must always be a doubly reversible, one of the number $\mathrm{R}^2(4x+1,2x)$; for the lines j, which disappear (Art. 3) in such reduction, are placed symmetrically about two axial planes: wherefore that symmetry must remain in the subject P to which the figures reduce themselves.

One of the axial planes (Art. 5) of P must contain k', one of the $4x-3$ edges k, which k' (Art. 13) can receive no point i, if the result is to be doubly reversible and all the summits triaces.

In any one of the quadrants between these planes will be seen $x-1$ edges k, on which $\frac{1}{4}m$ points i are to be planted, and the configurations are to be completed so that every quadrant shall read as the reverse of that on either side of it. The number of results will be,

$$\frac{\left(\frac{m}{4}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}} \cdot 2^{\frac{m}{4}};$$

wherefore

$$\mathrm{R}^2(4x+m+1,2x) = 2^{\frac{m}{4}} \cdot \frac{\left(\frac{m}{4}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}} \mathrm{R}^2(4x+1,2x),$$

which is the whole number $\mathrm{R}^2(4x+m+1,2x.)$

This vanishes if $x=1$, unless $m=0$, showing that the wedge is the only doubly reversible with two triangles.

16. PROBLEM *i*. To determine the number of doubly irreversible $(4x+m+1)$-edra reducible to doubly reversible $(4m+1)$-edra having $2x$ triangles.

Nothing prevents us here from depositing points *i* on the axial edge *k'* of the subject, inasmuch as the operations in one half of that edge can be repeated in the other half without introducing tessaraces.

On one side of the axial plane which bisects *k'* we see $2x-1$ edges *k*, including the half of *k'*, wherefore the number of ways of drawing $\frac{1}{2}m$ lines *j* is

$$\frac{\left(\frac{m}{2}+1\right)^{\overline{2x-2\,|\,1}}}{1^{\overline{2x-2\,|\,1}}} \cdot 2^{\frac{m}{2}},$$

which operations are to be repeated in direct order on the other side of the plane. This gives us only doubly irreversible results, but every one is twice obtained because of the reversible configuration on which we have operated. To every face *h* on one side of our bisecting plane corresponds another face *h'* on the same side, such that the sequence read from *h* in one direction is identical with that read from *h'* in the opposite; so that every sequence is twice constructed. We have therefore to divide the above by two; hence

$$\mathrm{I}^2\mathrm{R}^2(4x+m+1,2x) = \frac{\left(\frac{m}{2}+1\right)^{\overline{2x-2\,|\,1}}}{1^{\overline{2x-2\,|\,1}}} \cdot 2^{\frac{m-2}{2}} \cdot \mathrm{R}^2(4x+1,2x);$$

which is a portion of the number $\mathrm{I}^2(4x+m+1,2x)$.

No new results can be obtained by operating on the semi-solid cut off by the other bisecting plane; for if in P' a result so gained we move the bisecting plane to its old position in our subject, we see on one side of it a configuration before enumerated, since all possible ones have been before enumerated.

17. PROBLEM *j*. To determine the singly reversible $(4x$

$+m+1$)-edra reducible to a doubly reversible $(4x+1)$-edron having $2x$ triangles.

We can produce singly reversible configurations bisected by either of the axial planes of the doubly reversible subject. Taking first that plane which contains the axial edge k', we see on one side of our plane $2x-2$ edges k (Art. 13) on which we can deposit $\frac{1}{2}m$ points i, thus obtaining

$$\frac{\left(\frac{m}{2}+1\right)^{\overline{2x-3\,|\,1}}}{1^{\overline{2x-3\,|\,1}}}\cdot 2^{\frac{m}{2}}$$

results, which are to be repeated in order reversed on the other side of the bisecting plane.

Among these will occur once every doubly reversible $(4x+m+1)$-edron with $2x$ triangles enumerated under problem h (Art. 15); for if the bisecting plane be drawn through the axial edge k' of any of these, we see a configuration that we have been producing here; and evidently this configuration, and therefore this $(4x+m+1)$-edron has been only once here obtained. And every singly reversible $(4x+m+1)$-edron among our present results will occur twice, because of the reversible character of the semi-solid on which we have been operating, as pointed out in the preceding article. Moreover all these singly reversibles have an edge k' in the axial plane, and all the singly reversible $(4x+m+1)$-edra having an axial edge k' which are reducible to a doubly reversible have been here constructed. Therefore, subtracting the results of Art. 15 and dividing the remainder by two, we obtain

$$\mathrm{RR}^2(4x+m+1,2x)=\underset{kax=i}{\left\{\frac{\left(\frac{m}{2}+1\right)^{\overline{2x-3\,|\,1}}}{1^{\overline{2x-3\,|\,1}}}\cdot 2^{\frac{m-2}{2}}\right.}$$
$$\left.-\frac{\left(\frac{m}{4}+1\right)^{\overline{x-2\,|\,1}}}{1^{\overline{x-2\,|\,1}}}\cdot 2^{\frac{m-4}{4}}\right\}\mathrm{R}^2(4x+1,2x);$$

which is a portion of $\mathrm{R}(4x+m+1,2x)$. Here m is of necessity even, and the expression vanishes for $x=1$, if $m>0$.

18. We consider next the bisecting axial plane of our subject which cuts the axial edge k'. On either side of it we see $2x-1$ edges k, including half of k', on which, m being still supposed even, we have to deposit $\frac{1}{2}m$ points i. The number of results is

$$\frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m}{2}}.$$

By reversing these operations on the other side of the axial plane, we shall again produce, and once only, every result of Art. 15; for if in any of these the bisecting plane be drawn to cut the axial edge k', we shall see one of the configurations here constructed. And evidently this has been constructed only once by the process of this article. Also by the reasoning of the preceding article it appears that every singly reversible $(4x+m+1)$-edron here obtained has been twice obtained, and has no axial edge. Subtracting then and dividing by two as before, we get

$$\mathrm{RR}^2(4x+m+1,2x)=\sum_{kax=0}\left\{\frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m-2}{2}}\right.$$
$$\left.-\frac{\left(\frac{m}{4}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\cdot 2^{\frac{m-4}{4}}\right\}\mathrm{R}^2(4x+1,2x),$$

which is a part of the number $\mathrm{R}(4x+m+1,2x)$.
$kax=0.$

Suppose next that m is odd, the axial plane still cutting the edge k'. We can with the even number $m-1$ produce as before

$$\frac{\left(\frac{m-1}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m-1}{2}}$$

results. We can then deposit one more point i at the

centre of the edge k', in the axial plane, and from this draw either of the lines j, in every figure constructed.

But none of these constructions will be doubly reversible, being not the same on both sides the axial plane containing k'. If then we multiply the above by two, we shall have, as shown in the preceding article, double the number of singly irreversibles when m is odd, and no subtraction of doubly reversible results has to be made. That is,

$$\mathrm{RR}^2(4x+m+1,2x)=\mathrm{R}^2(4x+1,2x)\cdot\frac{\left(\frac{m-1}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m-1}{2}},$$

a portion of $\mathrm{R}(4x+m+1,2x)$. In these the base is odd-angled, being $(4x+m)$-gonal, and in these alone, having m odd, is there a line j in the axial plane, as denoted by the sub-index $jax=1$.

19. Problem k. To determine the number of singly irreversible $(4x+m+1)$-edra reducible to a doubly reversible $(4x+1)$-edron having $2x$ triangles.

We can distribute m points on the $4x-3$ edges k (Art. 7). This gives us

$$\frac{(m+1)^{4x-4\,|\,1}}{1^{4x-4\,|\,1}}\cdot 2^m$$

arrangements of lines j.

Among these results will occur every one of $\mathrm{R}^2\mathrm{R}^2$ (Art. 15) once, and once only; for the lines j in any of these can be laid on the subject in one way only: every one of $\mathrm{I}^2\mathrm{R}^2$ (Art. 16) will occur twice, in one result appearing as the reflected image of the other: every singly irreversible $(\mathrm{RR}^2)'$ will be constructed twice, viz. the same configuration being laid on both sides of that axial plane of the subject which is not the axial plane of $(\mathrm{RR}^2)'$. Every singly irreversible now to be enumerated will present itself four times; for any sequence read about the base of the subject of operation is read four times, beginning in every

quadrant, and being read in one direction in two quadrants and in another in the other two; wherefore every constructed configuration will be begun at four different points of the circuit, and appear in two results as the reflected image of itself seen in the other two.

If then from the above number of arrangements of lines we subtract the doubly reversibles of Art. 15, twice the doubly irreversibles of Art. 16, and twice the reversibles of 17 and 18, the remainder divided by four is the number of irreversibles here required. That is,

$$\begin{aligned}
\mathrm{IR}^2(4x+m+1,2x) = \tfrac{1}{4}\mathrm{R}^2(4x+1,2x)\Bigg\{ & \frac{(m+1)^{4x-4\,|\,1}}{1^{4x-4\,|\,1}}\cdot 2^m \\
& - \frac{\left(\frac{m}{4}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\, 2^{\frac{m}{4}} - 2\frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\, 2^{\frac{m-2}{2}} \\
& - 2\frac{\left(\frac{m}{2}+1\right)^{2x-3\,|\,1}}{1^{2x-3\,|\,1}}\cdot 2^{\frac{m-2}{2}} + \frac{\left(\frac{m}{4}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\cdot 2^{\frac{m}{4}} \\
& - 2\frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{(2x-2)\,|\,1}}\, 2^{\frac{m-2}{2}} + \frac{\left(\frac{m}{4}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\, 2^{\frac{m}{4}} \\
& - 2\frac{\left(\frac{m+1}{2}\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m-1}{2}}\Bigg\}, \\
= \mathrm{R}^2(4x+1,2x)\Bigg\{ & \frac{(m+1)^{4x-4\,|\,1}}{1^{4x-4\,|\,1}}\cdot 2^{m-2} + \frac{\left(\frac{m}{4}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\cdot 2^{\frac{m-8}{4}} \\
& - \frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m-2}{2}} - \frac{\left(\frac{m}{2}+1\right)^{2x-3\,|\,1}}{1^{2x-3\,|\,1}}\cdot 2^{\frac{m-4}{2}} \\
& - \frac{\left(\frac{m+1}{2}\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m-3}{2}}\Bigg\},
\end{aligned}$$

which is part of the number $\mathrm{I}(4x+m+1,2x)$.

20. Problem *l*. To determine the number of triply reversible $(6x+m+1)$-edra having $3x$ triangles.

All these by the definition of a triply reversible, Art. 5,

have $3x$ triangles. If these triangles all vanish, the result will have $3x'$ $(x' < x)$ triangles, and thus we shall at last reduce all the figures to a triply reversible having only 3 triangles, which is incapable of further reduction in the number of its triangles. Any axial plane bisecting the base must therefore pass through the vertex of a triangle, and bisect the triangle; it must therefore contain the edge k which passes through that vertex. That is, three of the edges not meeting the base are axial edges; consequently the polyedra considered in the problem have three axial edges, and are obtained by the process of our previous problems from $(6x+1)$-edra, having $3x$ triangles, and three axial edges that cannot receive points i. Taking any one of these $(6x+1)$-edra for the subject of our operations, we have $6x-6$ edges in which to plant m points i, *i.e.* $x-1$ edges in the sixth part of the circuit on which to lay $\frac{1}{6}m$ points i, as our result is to be (Art. 5) triply reversible. We obtain

$$\frac{\left(\frac{m}{6}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}} \cdot 2^{\frac{m}{6}}$$

results from every subject of operation, which are to be repeated in reversed order in all the six sextants between the axial planes so that all shall be triply reversible. Hence follows

$$\mathrm{R}^3\mathrm{R}^3(6x+m+1,3x) = \frac{\left(\frac{m}{6}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}} \cdot 2^{\frac{m}{6}} \cdot \mathrm{R}^3(6x+1,3x),$$

which vanishes for $x=1$, unless $m=0$, showing that no triply reversible with three triangles exists except the 7-edron with three triangles.

From Art. 15 it is evident that no doubly reversible can be thus generated from a triply reversible subject of operation.

21. Problem m. To determine the number of singly re-

versible $(6x+m+1)$-edra having $3x$ triangles that are reducible to triply reversible $(6x+1)$-edra having $3x$ triangles.

It is of no consequence about which of the three axial bisecting planes in our subject we operate, for all of them contain an axial edge k, and reduce by the vanishing of triangles to the tetraedron. We have $6x-4$ edges k on which to plant m points i, that is, $3x-2$ on one side of the axial plane, on which to lay $\frac{1}{2}m$ points i, for singly reversible results; operations which are to be reversed on the other side of the plane. Among the

$$\frac{\left(\frac{m}{2}+1\right)^{3x-3\,|\,1}}{1^{3x-3\,|\,1}}\,2^{\frac{m}{2}}$$

results obtained from our subject will occur every triply reversible of the preceding article, for they are all of course reversible configurations about our bisecting plane, and will occur only once, as will also every singly reversible. For as there is no axial plane at right angles to the one we operate on, the configuration of the subject reads differently from the two extremities of the axis of reversion in the base; *i.e.* we can only produce a given reversible result by one position of our m lines j on the subject. Wherefore, subtracting the triply reversibles,

$$\mathrm{RR}^3(6x+m+1,3x)$$

$$=\mathrm{R}^3(6x+1,3x)\left\{2^{\frac{m}{2}}\frac{\left(\frac{m}{2}+1\right)^{3x-3\,|\,1}}{1^{3x-3\,|\,2}}-2^{\frac{m}{6}}\frac{\left(\frac{m}{6}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\right\},$$

which is part of the number $\mathrm{R}(6x+m+1,3x.)_{kax=1}$.

22. Problem *n*. To determine the number of triply irreversible $(6x+m+1)$-edra having $3x$ triangles, which are reducible (Art. 3) to $(6x+1)$-edra triply reversible with $3x$ triangles.

We have $(6x-3)$ edges k on which to lay m points, or $2x-1$ in the third part of the circuit on which to lay $\frac{1}{3}m$

points for triply irreversible results. This can be done in

$$\frac{\left(\frac{m}{3}+1\right)^{\overline{2x-2\,|\,1}}}{1^{\overline{2x-2\,|\,1}}}\cdot 2^{\frac{m}{3}}$$

ways, which operations are to be twice again performed in the same order and direction in the other two thirds of the circuit of our subject. This gives us the result,

$$\mathrm{I^3R^3}(6x+m+1,3x)=\frac{\left(\frac{m}{3}+1\right)^{\overline{2x-2\,|\,1}}}{1^{\overline{2x-2\,|\,1}}}\cdot 2^{\frac{m}{3}}\cdot \mathrm{R^3}(6x+1,3x),$$

a portion of $\mathrm{I^3}(6x+m+1,3x)$.

23. Problem *o*. To determine the number of singly irreversible $(6x+m+1)$-edra having $3x$ triangles that are reducible (Art. 3) to triply irreversible $(6x+1)$-edra with $3x$ triangles.

If on the $6x-3$ edges k we deposit m points i, we obtain

$$\frac{(m+1)^{\overline{6x-4\,|\,1}}}{1^{\overline{6x-4\,|\,1}}}\cdot 2^m$$

results. Among these will occur every triply reversible of the number $\mathrm{R^3R^3}(6x+m+1,3x)$ (Art. 20) once, and once only; every reversible of $\mathrm{RR^3}(6x+m+1,3x)$ (Art. 21) will be found three times, being constructed about each of the bisecting axial planes of the subject; every figure of $\mathrm{I^3R^3}(6x+m+1,3x)$ of the preceding article will present itself twice, *i.e.* each figure and its reflected image once; and every singly irreversible here required will be seen six times, each figure and its reflection being constructed to have the same complete circuit begun at three different points of the triple subject.

From the entire results set down above we must, therefore, subtract the $\mathrm{R^3R^3}$, three times the $\mathrm{RR^3}$, and twice the $\mathrm{I^3R^3}$ of Art. 20, 21, 22, and divide the remainder by six. This gives us

$$\mathrm{IR^3}(6x+m+1,3x)=\tfrac{1}{6}\left\{\frac{(m+1)^{\overline{6x-4\,|\,1}}}{1^{\overline{6x-4\,|\,1}}}\cdot 2^m\right.$$

$$-\frac{\left(\frac{m}{6}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\cdot 2^{\frac{m}{6}}-3\frac{\left(\frac{m}{2}+1\right)^{3x-3\,|\,1}}{1^{3x-3\,|\,1}}\cdot 2^{\frac{m}{2}}$$
$$+3\frac{\left(\frac{m}{6}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\cdot 2^{\frac{m}{6}}-2\frac{\left(\frac{m}{3}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m}{3}}\Bigg\}$$
$$\times \mathrm{R}^3(6x+1,3x);\ \text{or}$$
$$\mathrm{IR}^3(6x+m+1,3x)=\tfrac{1}{8}\cdot\mathrm{R}^3(6x+\mathrm{I},3x)$$
$$\times\left\{\frac{(m+1)^{6x-4\,|\,1}}{1^{6x-4\,|\,1}}\cdot 2^m+\frac{\left(\frac{m}{6}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\cdot 2^{\frac{m+6}{6}}\right.$$
$$\left.-3\frac{\left(\frac{m}{2}+1\right)^{3x-3\,|\,1}}{1^{3x-3\,|\,1}}2^{\frac{m}{2}}-\frac{\left(\frac{m}{3}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m+3}{3}}\right\},$$

which is part of the number $\mathrm{I}(6x+m+1,3x)$.

24. We have now all the formulæ necessary for the enumeration of the triedral partitions of the r-ace, that is of the triangular partitions of the r-gon,—in other words, for the enumeration of the $(x+1)$-edra on an x-gonal base whose summits are all triedral, or of the $(x+1)$-acra having an x-ace and only triangular faces. At least we have all the formulæ except the following (Art. 4), P standing for polyedra,

$$\mathrm{P}(2x+1,x)=\mathrm{P}(x+1,2)+\mathrm{P}(x+1,3)+\mathrm{P}(x+1,4)$$
$$+\cdots+\mathrm{P}(x+1,\tfrac{1}{2}x),=\mathrm{P}(x+1),$$

which means that the $(2x+1)$-edra on $2x$-gonal base having triedral summits and x triangles, which includes all the subjects of operation in the preceding problem, are in number equal to the $(x+1)$-edra on x-gonal base, having triedral summits and two or more triangles, *i.e.* to the $(x+1)$-edra having an x-gonal base and triedral summits. When x is odd, the term $\mathrm{P}(x+1,\frac{1}{2}x)$ is zero, the series terminating with $\mathrm{P}(x+1,\frac{1}{2}(x-1))$. It is evident that the reversible or irreversible character of $\mathrm{P}(x+1,f)$ remains in each figure, when the x summits of the base are cut away.

25. Our results are now to be brought into one view, each with a reference to the article in which it is proved.

Observe that $m>o$ and that a factorial containing $\frac{m\pm e}{a}$ not integer is always zero.

(4). $\mathrm{P}(2x+1,x)=\mathrm{P}(x+1,2)+\mathrm{P}(x+1,3)+\mathrm{P}(x+1,4)+\cdot$
$=\mathrm{P}(x+1).$

(8). $\mathrm{II}(2x+m+1x)=\mathrm{I}(2x+1,x)\cdot 2^m\cdot\frac{(m+1)^{2x-4\,|\,1}}{1^{2x-4\,|\,1}}\cdot$

(9). $\mathrm{I^2I^2}(4x+m+1,2x)=\mathrm{I^2}(4x+1,2x)\cdot 2^{\frac{m}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}.$

(10). $\mathrm{II^2}(4x+m+1,2x)=\mathrm{I^2}(4x+1,2x)$
$$\times\left\{2^{m-1}\cdot\frac{(m+1)^{4x-4\,|\,1}}{1^{4x-4\,|\,1}}-2^{\frac{m-2}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\right\}.$$

In (9) and (10) m is always even.

(11). $\mathrm{I^3I^3}(6x+m+1,3x)=\mathrm{I^3}(6x+1,3x)\cdot 2^{\frac{m}{3}}\cdot\frac{\left(\frac{m}{3}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}};$
$(m=3n).$

(12). $\mathrm{II^3}(6x+m+1,3x)=\frac{1}{3}\mathrm{I^3}(6x+1,3x)$
$$\times\left\{2^m\cdot\frac{(m+1)^{6x-4\,|\,1}}{1^{6x-4\,|\,1}}-2^{\frac{m}{3}}\cdot\frac{\left(\frac{m}{3}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\right\}.$$

(13). $\mathrm{RR}(2x+m+1,x)=\underset{kax=1.}{\mathrm{R}}(2x+1,x)\cdot 2^{\frac{m}{2}}\cdot\underset{kax=0.}{\frac{\left(\frac{m}{2}+1\right)^{x-3\,|\,1}}{1^{x-3\,|\,1}}},$

$$\mathrm{RR}(2x+m+1,x)=\underset{kax=0.}{\mathrm{R}}(2x+1,x)\cdot 2^{\frac{m}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}},$$

$$\mathrm{RR}(2x+m+1,x)=\underset{jax=1.}{\mathrm{R}}(2x+1,x)\cdot 2^{\frac{m+1}{2}}\cdot\underset{kax=0.}{\frac{\left(\frac{m+1}{2}\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}},$$

(14). $\mathrm{IR}(2x+m+1,x)=\frac{1}{2}\mathrm{R}(2x+1,x)\cdot 2^m\cdot\frac{(m+1)^{2x-4\,|\,1}}{1^{(2x-4)\,|\,1}},$

$$-\frac{1}{2}\underset{kax=1.}{\mathrm{R}}(2x+1,x)\times 2^{\frac{m}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{x-3\,|\,1}}{1^{x-3\,|\,1}}$$

$$- \left\{ 2^{\frac{m}{2}} \cdot \frac{\left(\frac{m}{2}+1\right)^{x-1|1}}{1^{x-2|1}} + 2^{\frac{m+1}{2}} \cdot \frac{\left(\frac{m+1}{2}\right)^{x-2|1}}{1^{x-2|1}} \right\}$$

$$\times \tfrac{1}{2} \mathrm{R}(2x+1,x)_{kax=0}.$$

(15). $\mathrm{R}^2(4x+m+1,2x) = \mathrm{R}^2(3x+1,2x) \cdot 2^{\frac{m}{2}} \cdot \frac{\left(\frac{m}{4}+1\right)^{x-2|1}}{1^{x-2|1}}$

(16). $\mathrm{I}^2\mathrm{R}^2(4x+m+1,2x) = \mathrm{R}^2(4x+1,2x)$

$$\times 2^{\frac{m-2}{2}} \cdot \frac{\left(\frac{m}{2}+1\right)^{2x-2|1}}{1^{2x-2|1}}.$$

(17). $\mathrm{RR}^2(4x+m+1,2x) = \mathrm{R}^2(4x+1,2x)_{kax=1.}$

$$\times \left\{ 2^{\frac{m-2}{2}} \cdot \frac{\left(\frac{m}{2}+1\right)^{2x-3|1}}{1^{2x-3|1}} - 2^{\frac{m-4}{4}} \cdot \frac{\left(\frac{m}{4}+1\right)^{x-2|1}}{1^{x-2|1}} \right\}.$$

(18). $\mathrm{RR}^2(4x+m+1,2x) = \mathrm{R}^2(4x+1,2x)_{kax=0.}$

$$\times \left\{ 2^{\frac{m-2}{2}} \cdot \frac{\left(\frac{m}{2}+1\right)^{2x-2|1}}{1^{2x-2|1}} - 2^{\frac{m-4}{4}} \cdot \frac{\left(\frac{m}{4}+1\right)^{x-2|1}}{1^{x-2|1}} \right\}.$$

$\mathrm{RR}^2(4x+m+1,2x) = \mathrm{R}^2(4x+1,2x)_{jax=1.}$

$$\times 2^{\frac{m-1}{2}} \cdot \frac{\left(\frac{m+1}{2}\right)^{2x-2|1}}{1^{2x-2|1}}; \quad (m \text{ odd}).$$

(19). $\mathrm{IR}^2(4x+m+1,2x) = \mathrm{R}^2(4x+1,2x)$

$$\times \left\{ 2^{m-2} \cdot \frac{(m+1)^{4x-4|1}}{1^{4x-4|1}} + 2^{\frac{m-3}{4}} \cdot \frac{\left(\frac{m}{4}+1\right)^{x-2|1}}{1^{x-2|1}} \right.$$

$$- 2^{\frac{m-2}{2}} \cdot \frac{\left(\frac{m}{2}+1\right)^{2x-2|1}}{1^{2x-2|1}} - 2^{\frac{m-4}{2}} \cdot \frac{\left(\frac{m}{2}+1\right)^{2x-3|4}}{1^{2x-3|1}}$$

$$\left. - 2^{\frac{m-3}{2}} \cdot \frac{\left(\frac{m+1}{2}\right)^{2x-2|1}}{1^{2x-2|1}} \right\}.$$

(20). $\mathrm{R}^3(6x+m+1,3x) = \mathrm{R}^3(6x+1,3x) \cdot 2^{\frac{m}{6}} \cdot \frac{\left(\frac{m}{6}+1\right)^{x-2|1}}{1^{x-2|1}}.$

(21). $\mathrm{RR}^3(6x+m+1,3x)=\mathrm{R}^3(6x+1,3x)$
$$\times\left\{2^{\frac{m}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{\overset{kax-1.}{3x-3\,|\,1}}}{1^{3x-3\,|\,1}}-2^{\frac{m}{6}}\frac{\left(\frac{m}{6}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\right\}.$$

(22). $\mathrm{I}^3\mathrm{R}^3(6x+m+1,3x)=\mathrm{R}^3(6x+1,3x)$
$$\times 2^{\frac{m}{3}}\cdot\frac{\left(\frac{m}{3}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}.$$

(23). $\mathrm{IR}^3(6x+m+1,3x)=\frac{1}{3}\mathrm{R}^3(6x+1,3x)$
$$\times 2^{m-1}\cdot\frac{(m+1)^{6x-4\,|\,1}}{1^{6x-4\,|\,1}}+2^{\frac{m}{6}}\frac{\left(\frac{m}{6}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}$$
$$-3\cdot 2^{\frac{m-2}{2}}\cdot\frac{\left(\frac{m}{2}+1^{3x-3\,|\,1}\right.}{1^{3x-3\,|\,1}}-2^{\frac{m}{3}}\cdot\frac{\left(\frac{m}{3}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\Bigg\}.$$

26. If now we desire to enumerate $\mathrm{P}(x+1)$, the $(x+1)$-edra on an x-gonal base having triedral summits, or, in other words, to enumerate the triangular partitions of the x-gon; we write first,

(24). $\mathrm{P}(x+1)=\mathrm{P}(x+1,2)+\mathrm{P}(x+1,3)+\cdot$

(25). $\mathrm{P}(x+1,f)=\mathrm{R}^3(x+1,f)+\mathrm{R}^2(x+1,f)+\mathrm{R}(x+1,f)$
$+\mathrm{I}^3(x+1,f)+\mathrm{I}^2(x+1,f)+\mathrm{I}(x+1,f)$.

$\mathrm{R}^3(x+1,f)=\mathrm{R}^3(2f+(x-2f)+1,f)$ is given by (20);

$\mathrm{R}^2(x+1,f)=\mathrm{R}^2(2f+(x-2f)+1,f)$, by (15);

$\mathrm{R}(x+1,f)=(\mathrm{RR}+\mathrm{RR}^2+\mathrm{RR}^3)(2f+(x-2f)+1,f)$, . . by (13, 17, 18, 21).

$\mathrm{I}(x+1,f)=(\mathrm{II}+\mathrm{II}^2+\mathrm{II}^3+\mathrm{IR}+\mathrm{IR}^2+\mathrm{IR}^3)(2f+(x-2f)+1,f)$, by (8, 10, 12, 14, 19, 23);

$\mathrm{I}^2(x+1,f)=(\mathrm{I}^2\mathrm{I}^2+\mathrm{I}^2\mathrm{R}^2)(2f+(x-2f)+1,f)$, by (9) and (16);

$\mathrm{I}^3(x+1,f)=(\mathrm{I}^3\mathrm{I}^3+\mathrm{I}^3\mathrm{R}^3)(2f+(x-2f)+1,f)$, by (11) and (22).

And thus $\mathrm{P}(x+1)$ can be found in terms of

$$\Sigma_f\mathrm{P}(2f+1,f)=\Sigma_f(\mathrm{R}^3+\mathrm{R}^2+\mathrm{R}+\mathrm{I}^3+\mathrm{I}^2+\mathrm{I})(2f+1,f),$$

where f has every value from 2 to the integer next below $\frac{1}{2}x$.

We may either consider $P(x+1,f)$ to denote the triedral partitions of the x-ace which leave f pairs of contiguous edges of the x-ace undisturbed, or to denote the triangular partitions of the x-gon, which leave f angles of the x-gon untouched by a diagonal.

27. To show the use of these formulæ, we can begin at the beginning, thus:

$P(4)=1$;

a. $P(5)=R^2(5,2)=1$: $I^m(5,2)=0=R(5,2)=R^3(5,2)$.

b. $R^2(5+m,2)=0$, if $m>0$, by $(15)=0^m$.

c. $I^2(5+m,2=2^{\frac{m-2}{2}}$, by 16 and *a*).

d. $R(5+m,2)=0$, by (18.)
$kax=1.$

d'. $R(5+m,2)=2^{\frac{m-2}{2}}$, by (18 and *a*); m even.
$kax=0.$

d''. $R(5+m,2)=2^{\frac{m-1}{2}}$, by (18 and *a*); m odd.
$jax=1.$

e. $I(5+m,2)=2^{m-2}-2^{\frac{m-2}{2}}-2^{\frac{m-3}{2}}$, by (19 and *a*); in which, of course, $m>1$ and either the second or third term must $=0$, as m is odd or even.

As $I^3(5+m,2)=R^3(5+m,2)$ evidently $=0$, we have, by addition of *c*, *d*, *d'*, *d''*, *e*,

A. $P(5+m,2)=I^2(5+m,2)+R(5+m,2)+I(5+m,2)$
$=2^{m-2}+2_m\cdot 2^{\frac{m-2}{2}}+2_{m-1}\cdot 2^{\frac{m-3}{2}}$,

where $m>0$, and $2_m=1-2_{m-1}$ is a circulator of the usual form; *i.e.* $s_m=1$ or $=0$, as $\frac{m}{s}$ is or is not integer.

This $P(5+m,2)$ is the number of triangular partitions of the $(5+m)$-gon in which two angles, and two only, are untouched by diagonals, or the number of triedral partitions of the $(5+m)$-ace in which two pairs of rays, and two only, of the x-ace remain each a united pair.

f. $P(7,3)=P(4)=R^3(7,3)=1$, by (4,25).

g. $R^3(7+m,3)=0^m$, by (20).

h. $RR^3(7+m,3)=2^{\frac{m}{3}}$, by (21,*f*), $(m>0)$.

i. $I^3R^3(7+m,3)=2^{\frac{m}{3}}$, by (22,*f*).

j. $IR^3(7+m,3)=\frac{1}{6}\left\{\frac{(m+1)^{2|1}}{1^{2|1}}\cdot 2^m-3\cdot 2^{\frac{m}{2}}-2^{\frac{m}{3}}\right\}$, by (23,*f*).

$(h+i+j)$ gives,

B. $P(7+m,3)=\frac{(m+1)(m+2)}{3}\cdot 2^{m-2}+2_m\cdot 2^{\frac{m-2}{2}}+3_m\cdot \frac{1}{3}\cdot 2^{\frac{m}{3}}$, $(m>0)$;

the number of triedral partitions of the $(7+m)$-ace in which three pairs of rays remain each a united pair, or of triangular partitions of the $(7+m)$-gon in which three triangles are marginal.

k. $P(9,4)=P(5)=R^3(5,2)=R^2(9,4)=1$, by (4,*a*,25).

l. $R^2(9+m,4)=2^{\frac{m}{4}}$, $(m>0)$, by (15,*k*).

m. $I^2R^2(9+m,4)=\frac{\left(\frac{m}{2}+1\right)^{2|1}}{1^{2|1}}\cdot 2^{\frac{m-2}{2}}-2^{\frac{m-4}{4}}$, by (16,*k*).

n. $RR^2(9+m+4)_{kax=1.}=\left(\frac{m}{2}+1\right)\cdot 2^{\frac{m-2}{2}}-2^{\frac{m-4}{4}}$, by (17,*k*).

n'. $RR^2(9+m,4)_{kax=0.}=\frac{\left(\frac{m}{2}+1\right)^{2|1}}{1^{2|1}}\cdot 2^{\frac{m-2}{2}}-2^{\frac{m-4}{4}}$, by (18,*k*).

n''. $RR^2(9+m,4)_{jax=1.}=\frac{\left(\frac{m+1}{2}\right)^{2|1}}{1^{2|1}}\cdot 2^{\frac{m-1}{2}}$ (m odd), by (18,*k*).

o. $IR^2(9+m,4)=\frac{(m+1)^{4|1}}{1^{4|1}}\cdot 2^{m-2}+2^{\frac{m-3}{4}}-\frac{\left(\frac{m}{2}+1\right)^{2|1}}{1^{2|1}}\cdot 2^{\frac{m-2}{2}}-\left(\frac{m}{2}+1\right)\cdot 2^{\frac{m-4}{2}}-\frac{\left(\frac{m+1}{2}\right)^{2|1}}{1^{2|1}}\cdot 2^{\frac{m-3}{2}}$, by (19,*k*).

The addition of $(l+m+n+n'+n''+o)$, remembering the observation (Art. 25), gives,

C. $$P(9+m,4)=\frac{(m+1)^{4|1}}{1^{4|1}}\cdot 2^{m-2}-4_m\cdot 2^{\frac{m-3}{4}}$$
$$+2_m\cdot(m+2)(m+6)\cdot 2^{\frac{m-3}{2}}$$
$$+2_{m-1}\cdot(m+1)(m+3)\cdot 2^{\frac{m-2}{2}},\ m>0);$$

the number of triedral partitions of the $(9+m)$-ace having four pairs of rays united pairs, or of the triangular partitions of the $(9+m)$-gon having four marginal triangles.

p. $$P(11,5)+P(6)=R(6,2)_{jax=1.}=1=R(11,5)_{kax=1.}\ (4,d'');$$

for when every base summit of R(6,2) is cut, as in Art. 4, the line j in the axis becomes a line k in the axis.

q. $$RR(11+m,5)_{kax=1.}=\frac{\left(\frac{m}{2}+1\right)^{2|1}}{1^{2|1}}\cdot 2^{\frac{m}{2}},\ \text{by (13 and } p).$$

r. $$IR(11+m,5)=\frac{(m+1)^{6|1}}{1^{6|1}}\cdot 2^{m-1}-\frac{\left(\frac{m}{2}+1\right)^{2|1}}{1^{2|1}}\cdot 2^{\frac{m-2}{2}},$$
by (14,*p*).

The addition of these gives us,

D. $$P(11+m,5)=\frac{(m+1)^{6|1}}{1^{6|1}}\cdot 2^{m-1}$$
$$+2_m\cdot(m+2)(m+4)\cdot 2^{\frac{m-3}{2}},\ (m>0;$$

the number of triedral partitions of the $(11+m)$-ace having five of the angles about the vertex undisturbed, or of the triangular partitions of the $(11+m)$-gon which have five marginal triangles.

s. $$P(13,6)=P(7,2)+P(7,3)$$
$$=I^2(7,2)+R(7,2)_{kax=c.}+R^2(7,3)$$
$$=1+1+1,\ \text{by } (4,c,d',g),$$
$$=I^2(13,6)+R(13,6)_{kax=0.}+R^2(13,6),\ \text{(Art. 4)}.$$

t. $$I^2I^2(13+m,6)=2^{\frac{m}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{4|1}}{1^{4|1}},\ \text{by } (9,s).$$

u. $\cdot \mathrm{II}^3(13+m,6)=2^{m-1}\cdot\frac{(m+1)^{8|1}}{1^{8|1}}-2^{\frac{m-2}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{4|1}}{1^{4|1}}$,
by (10,*s*).

v. $\mathrm{RR}(13+m,6)\underset{kax=0.}{=}2^{\frac{m}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{4|1}}{1^{4|1}}$, by (13,*s*).

w. $\mathrm{RR}(13+m,6)\underset{jax=1.}{=}2^{\frac{m+1}{2}}\cdot\frac{\left(\frac{m+1}{2}\right)^{4|1}}{1^{4|1}}$, by (13,*s*).

x. $\mathrm{IR}(13+m,6)=2^{m-1}\cdot\frac{(m+1)^{8|1}}{1^{8|1}}-2^{\frac{m-2}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{4|1}}{1^{4|1}}$
$-2^{\frac{m-1}{2}}\cdot\frac{\left(\frac{m+1}{2}\right)^{4|1}}{1^{4|1}}$, by (14,*s*).

y. $\mathrm{R}^3(13+m,6)=6_m\cdot 2^{\frac{m}{6}}$, by (20,*s*).

z. $\mathrm{RR}^3(13+m,6)\underset{kax=1.}{=}2^{\frac{m}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{3|1}}{1^{3|1}}-6_m\cdot 2^{\frac{m}{6}}$, by (21,*s*).

aa. $\mathrm{I}^3\mathrm{R}^3(13+m,6)=2^{\frac{m}{3}}\cdot\frac{\left(\frac{m}{3}+1\right)^{2|1}}{1^{2|1}}$, by (22,*s*).

bb. $\mathrm{IR}^3(13+m,6)=\frac{1}{3}\left\{2^{m-1}\cdot\frac{(m+1)^{8|1}}{1^{8|1}}+2^{\frac{m}{6}}\right.$
$\left.-3\cdot 2^{\frac{m-2}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{3|1}}{1^{3|1}}-2^{\frac{m}{3}}\cdot\frac{\left(\frac{m}{3}+1\right)^{2|1}}{1^{2|1}}\right\}$,
by (23,*s*).

The addition of these nine quantities gives us,

E. $\mathrm{P}(13+m,6)=\frac{1}{3}\cdot 2^{m-1}\cdot\frac{(m+1)^{8|1}}{1^{8|1}}$
$+6_m\cdot\frac{1}{3}\cdot 2^{\frac{m}{6}}+3_m\cdot\frac{1}{3}\cdot 2^{\frac{m+3}{3}}\cdot\frac{\left(\frac{m}{3}+1\right)^{2|1}}{1^{2|1}}$
$+2_m\left\{2^{\frac{m}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{4|1}}{1^{4|1}}+2^{\frac{m-2}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{3|1}}{1^{3|1}}\right\}$

$$+2_{m-1}\cdot\frac{\left(\frac{m+1}{2}\right)^{4\,|\,1}}{1^{4\,|\,1}}\cdot 2^{\frac{m-1}{2}};$$

the number of triedral partitions of the $(13+m)$-ace having six angles of the polyace undisturbed, or that of the triangular partitions of the $(13+m)$-gon in which are always six marginal triangles.

cc. $P(15,7)=P(8,2)+P(8,3)$

$=R(8,2)+I(8,2)+I(8,3)$

$jax-1.$

$=2+1+1$, by (d'',e,j)

$=R(15,7)+I(15,7)$

$kax-1.$

$=2+2.$ (Vide *p*, and Art. 4.)

dd. $RR(15+m,7)=2\cdot\frac{\left(\frac{m}{2}+1\right)^{4\,|\,1}}{1^{4\,|\,1}}\cdot 2^{\frac{m}{2}}$, by (13,*cc*).

$kax-1.$

ee. $IR(15+m,7)=1\cdot\frac{(m+1)^{10\,|\,1}}{1^{10\,|\,1}}\cdot 2^m-1\cdot\frac{\left(\frac{m}{2}+1\right)^{4\,|\,1}}{1^{4\,|\,1}}\cdot 2^{\frac{m}{2}}$,

by (14,*cc*).

ff. $II(15+m,7)=2\cdot 2^m\cdot\frac{(m+1)^{10\,|\,1}}{1^{10\,|\,1}}$, by (8,*cc*).

Adding these three quantities, we obtain,

F. $P(15+m,7)=3\cdot 2^m\cdot\frac{(m+1)^{10\,|\,1}}{1^{10\,|\,1}}+2_m\cdot 2^{\frac{m}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{4\,|\,1}}{1^{4\,|\,1}}$;

the number of these partitions of the $(15+m)$-ace, or of the $(15+m)$-gon.

And thus we can, without difficulty, proceed to find for any value of x and m, $P(2x+m+1,x)$, the triedral partitions of the $(2x+m+1)$-ace having x pairs of rays united pairs, or the triagular partitions of the $(2x+m+1)$-gon having x marginal triangles.

In the *Philosophical Transactions for* 1856, in a memoir "On the enumeration of the x-edra having triedral summits and an $(x-1)$-gonal base," I have given a

solution of the problem of this paper in a form less calculated for obtaining easily the algebraic expression of $P(2x+m+1,x)$; and in the same *Transactions for* 1857, in a memoir "On the partitions of the polygon and the polyace," I have given formulæ for all these partitions, whether triangular or not. The method of this paper is unquestionably the best adapted for the deduction of the general expression of these partitions, and is capable of being extended to them all.

IV. — *On Improvement in Meteorological Registration.*

By Mr. Thomas Hopkins.

Read January 12th, 1858.

Great numbers of persons are engaged in collecting information respecting the changes which, it is well known, frequently take place in our atmosphere. Many individuals register the alterations indicated by meteorological instruments with great care and industry, and governments have established and maintained observatories for this purpose, at considerable cost; yet it has been said by eminent men, of whose competency to form opinions on the subject no doubt can exist, that the results obtained are of little value. This suggests the idea, that the course pursued in investigating the subject may be defective, and warrants a particular examination of the methods adopted.

The weight of the atmosphere, in each locality, is almost constantly varying, and the amount of that weight as well as the alterations which it undergoes are indicated by the barometer. But the most vigilant watching of the height of the mercury, combined with observations of the state of the sky, has failed to exhibit to our view the causes of the fluctuations of atmospheric pressure. There is a general impression that, in some way, heat lightens the atmosphere, and that cold makes it heavy, but in what way the changes, from the one to the other state, are effected, is not known.

The principal instruments used in meteorological investigations are the barometer, which may be considered to measure correctly the various alterations that occur in the amount of atmospheric pressure, the dry thermometer, and the wet bulb thermometer. In addition to these the dew-point, or point of condensation of vapour is ascertained, from which is deduced the tension, or pressure, of vapour. The barometer shows the total atmospheric pressure, but that pressure is known to be made up of two parts,—one the pressure of the gases, and the other that of the vapour which is diffused through them; and in investigating the subject it is necessary that we should know how the separate pressures of their constituents are ascertained, in order that we may judge of the degree of reliance that may be placed on the results that are obtained.

It has been found by experiment that aqueous vapour loses its aeriform state, and is converted into water, when exposed to a sufficient degree of cold, or, in other words, when heat is sufficiently abstracted from it by reduction of temperature. And the more dense the vapour, the more readily does a reduction of temperature convert a part of the vapour into liquid. Hence it is seen that the varying quantities of aqueous vapour found in the atmosphere may be condensed by different degrees of cold; and the particular temperature at which the vapour is converted into water indicates the degree of abundance, or the density, in which it exists in any part of it. The temperature of liquefaction, or the dew-point, therefore shows the quantity of vapour present in the atmosphere. And each quantity has a particular weight and elastic force, with which it presses on the earth, and of course on the mercury of any barometer placed on its surface. This force is generally called the tension of vapour, and it indicates the quantity of vapour in the atmosphere, at any and every period of time.

The tension of vapour is found to be different in various climates and seasons. In warm and moist tropical countries where vapour exists in great abundance, its tension is sometimes equal in force to the weight of about an inch of mercury. In colder countries there is less vapour in the air, and therefore its tension is less, until in very cold and dry countries the quantity is so small as to make it a difficult task to ascertain its amount. But whatever may be the tension of the vapour that exists in the atmosphere, it is considered to constitute a part of the general atmospheric pressure which is measured by the barometer; and, the vapour pressure being deducted from the total atmospheric pressure, the remainder is deemed to belong to the gases. Tables of the tension of vapour have been constructed to show what belongs to each dew-point, by which means the separate pressure of vapour has been given in meteorological registrations. This being subtracted from general atmospheric pressure, as shown by the barometer, gives the separate amount of gaseous pressure. The barometer shows that atmospheric pressure alters, from time to time, to the extent of three inches of mercury, making one-tenth of the total average weight of the atmosphere; and as tension of vapour seldom equals one inch, the cause of the greater part of the alterations in the weight of the atmosphere must exist in changes of the gases; hence the importance of ascertaining what those changes are.

After finding what was due to tension of vapour, in order to account for the alterations which take place in the heights of the barometer, the temperatures of the gases, as ascertained by a thermometer, have been carefully given. It is well known that heat expands gases in the open atmospheric space; and in the proportion in which they expand they diffuse themselves around to a distance, becoming lighter in the part from which they

have expanded, and consequently pressing there with less weight on the earth and on any barometer near it.

Change in the temperature of the gases is therefore recognised as a cause of alteration of atmospheric pressure, as it is measured by the barometer, and, taken in conjunction with changes of vapour tension, is presumed by meteorologists in general, to cause all the fluctuations of the barometer, no other cause being acknowledged to influence atmospheric pressure.

Yet the barometer occasionally sinks without either temperature or vapour tension showing alterations to account for it. Indeed sometimes, tension of vapour varies but little, and temperature declines; but the barometer, so far from rising, as might be expected from the recognised influences of these two forces, actually falls, often to a considerable extent. From these circumstances we may infer that *neither reduction of vapour tension, as indicated by the dew-point, nor fall of the temperature of the gases, as shown by the thermometer near the surface, nor both together, are the real causes of the sinkings of the barometer that sometimes take place.*

The sun is undoubtedly the great source of the heat that disturbs the atmosphere, but its heating effect in different latitudes is unequal, and its influence on different parts varies also with its passage across the tropics; yet any partial effect thus produced on the weight of the atmosphere is, in no long time, counteracted and corrected by the operation of gravity, which immediately begins to act on the aerial ocean to restore the equilibrium of atmospheric pressure. Therefore no change of temperature as shown by the thermometer has furnished reasonable ground for believing that the great falls of the barometer that often occur can have been caused by such changes as are indicated by the thermometer.

The sun has a daily, as well as an annual influence, but

the effects of daily changes may be traced better than those of annual ones. The atmosphere is disturbed by solar influence every hour of the day, and these disturbances take place so regularly as to admit of their being followed and minutely examined, with a view to discover the precise way in which the solar influence is exerted. The daily changes therefore afford better means than the annual ones of inquiring into the causes of atmospheric disturbance. In most parts of the world, the atmosphere becomes heavier from four until ten o'clock in the morning, during which time the barometer rises; and the causes that must then have been in in action to produce these effects, being so regular, may be followed, and the forces may possibly be exhibited. This has been attempted for a considerable time, but the results obtained are by many persons deemed unsatisfactory; it therefore becomes desirable that we should examine the means that have been employed to account for barometric movements, and if they should be found to have been defective, to point out their defects and suggest improvements.

With the approach of the sun every morning, evaporation of water from the part of the earth's surface heated by it becomes more active, and additional aqueous vapour is sent into the air, showing itself in a higher dew-point and an increase of atmospheric pressure, which of course raises the mercury of the barometer. The hours during which these phenomena occur vary to a certain extent with latitude and season, but the average time is from four to ten o'clock in the morning, and within that period the barometer rises in consequence of an increase of aqueous vapour. At a certain time, however, say ten o'clock, the barometer begins to fall, although vapour continues to pass, even in greater abundance, into the atmosphere. In the present state of our knowledge of the subject we naturally ask, —What can be the cause

of this fall? Meteorologists point to the movements of the thermometer, and to the changes in tension of vapour, and endeavour to find in them an adequate cause for the reduction of atmospheric pressure which then occurs. But the registrations of their instruments have been of such a kind as to make this reference unsatisfactory, and have consequently permitted vague and insufficient explanations to be given by some, and to be received by others. These explanations have proceeded on the assumption that the fall of the barometer was to be accounted for by the alterations which take place in temperature and in vapour pressure, though this assumption may from many of these registers be shown to be erroneous. Of these, the observations made at Toronto in the year 1846, and published by the government, may be deemed the most suitable for our present purpose.

The whole Toronto observations extend over the years 1846-7-8; but in order to have a case sufficiently simple to admit of being dealt with in the present paper, while it is a fair specimen of what occurs in many parts of the world, I will extract the hourly registrations of the barometer from four o'clock in the morning until ten at night of each day during the month of July in the year 1846, and then give in the same tabular form, for the same time, the recorded amount of tension of vapour, as deduced from the dew-point. The tension of vapour being then subtracted from the total atmospheric pressure as shown by the barometer, the remainder corresponds, according to the present views of meteorologists, to the separate pressure of the atmospheric gases, which is also given in another line. Having appended the corresponding temperatures as recorded from a standard thermometer, we may proceed to examine how far the results harmonise with the views of meteorologists. — (*See the Table.*)

Hours.	A.M. 4	5	6	7	8	9	10	11	12	P.M. 1	2	3	4	5	6	7	8	9	10
BAROMETER.																			
29 inches	·5831	·5941	·5999	·6067	·6093	·6084	·6079	·6043	·5965	·5851	·5765	·5711	·5678	·5588	·5600	·5631	·5691	·5853	·5872
TENSION OF VAPOUR.																			
	·4420	·4430	·4880	·5310	·5450	·5550	·5500	·5360	·5440	·5300	·5450	·5330	·5190	·5190	·5230	·5190	·4980	·4800	·4640
GASEOUS PRESSURE.																			
29 inches	·1411	·1511	·1119	·0757	·0643	·0534	·0579	·0683	·0525	·0551	·0315	·0381	·0488	·0398	·0370	·0441	·0711	·1053	·1232
THERMOMETER.																			
	58°·89	59°·02	62°·63	66°·75	69°·24	71°·32	73°·17	73°·81	75°·05	74°·59	75°·65	75°·73	76°·07	75°·29	73°·22	70°·28	66°·66	63°·97	63°·20
WET BULB THERMOMETER.																			
	56°·88	57°·00	60°·06	63°·12	64°·52	65°·51	65°·73	65°·80	66°·44	65°·97	66°·78	66°·99	66°·51	65°·84	65°·26	64°·03	61°·98	60°·38	59°·39
EVAPORATION.																			
	2°·01	2°·02	2°·57	3°·63	4°·72	5°·81	8°·44	8°·01	8°·61	8°·62	8°·87	9°·34	9°·56	9°·45	7°·96	6°·25	4°·68	3°·59	3°·63

Here we may perceive that at four o'clock in the morning, the barometer stood at 29·5831 inches, and that by nine in the morning it had risen to 29·6084, being a rise of 0·0253 of an inch. At the first named hour, the tension of vapour was 0·4420, and by nine it had risen to 0·5550, or 0·1130 of an inch, leaving for the separate pressure of the gases at the first hour 29·1411 inches, from which it declined to 29·0534 at nine; so that the increase in the pressure of vapour must have counteracted the whole influence of increased temperature during the time, and also raised the barometer 0·0253 of an inch. The additional vapour pressure must have been a result of greater energy of evaporation, arising from the higher temperature sending more vapour into the air, and the increased evaporation is shown in the difference in the registrations of the wet and dry thermometers, as seen in the table above. This amounted to 2°·01 at four o'clock, and increased to 5°·81 by nine o'clock; and we have seen that up to that hour enough vapour had been sent into the air to raise the barometer 0·0253 of an inch, as well as to counteract the influence of the increased temperature on the gases.

A change now took place, and after nine o'clock the barometer fell until five in the afternoon, when it was as low as 29·5588 inches; and here we naturally ask: What was, at this time, the influence of the aqueous vapour which had just before shown itself so powerful in producing a rise in the barometer? It would seem to follow as a necessary consequence that as a rise of that instrument was caused by an increase of vapour, a fall would be accompanied by a decrease, or at least that the tension of vapour would show a reduction when the barometer fell equal to its increase when that instrument rose.

At nine o'clock, the time when the barometer began to fall, we have seen that evaporation was at 5°·81, but

instead of then becoming less, it became more energetic, and increased until five o'clock in the afternoon, when it was at 9°·45, making an increase in the energy of evaporation from nine in the morning until five in the afternoon of 3°·64, yet, notwithstanding this increase of evaporation, the barometer fell no less than 0·0496. It is, however, to be noticed, as may be seen from the table, that though evaporation had so greatly increased, the tension of vapour was somewhat reduced after nine o'clock; yet that reduction was small compared with the fall of the barometer. The tension was found at nine o'clock to be 0·5550, from which it became reduced until at five it was only 0·5190, but the barometer fell during the same time from 29·6084 to 29·5588. Thus it required an increase of 0·1130 of vapour tension in the morning to raise the barometer 0·0253; but a fall of that instrument afterwards until five in the afternoon of 0·0569 was accompanied by a reduction of only 0·0360 of tension. So that tension of vapour is exhibited as being powerful compared with temperature, the other varying constituent of pressure, in the morning, and feeble in the evening, involving a mass of contradiction and confusion.

From these statements, however, it becomes sufficiently apparent that the weights of the constituents of atmospheric pressure cannot be inferred from the tension of vapour, seeing that the vapour which passes into the atmosphere between nine and five does not appear in the tension. I have shown elsewhere how the vapour that passes into the air at this time of the day reduces, instead of increasing, tension, but the aqueous matter in the atmosphere must evidently be increased by the evaporation; though there are no means of measuring the extent of that increase, unless we ascertain directly the quantities of water evaporated. Yet whatever the quantity may be, when it does not fall as rain, it remains in the air either as vapour or as cloud. And while in the air as cloud, it does

not lose its property of gravitation, but as ponderable matter must float in and be sustained by the gases. Indeed, it may generally be seen there as mist or cloud, and as both are sustained by the gases, the weight and pressure of the gases must be augmented.

But from nine to five we have the fact that the barometer fell considerably, which proves that the gases must then have pressed on the surface with diminished force: and as they bear the weight of the clouds that float in them, they must themselves, from some cause, have been rendered so much lighter, as to be less heavy with the cloud in them than they were before the cloud was formed. That they are reduced in weight, whatever may be the cause of the reduction, is proved both by the falling of the barometer at the time, and by the kind of disturbance which takes place in the atmosphere. Solar heat makes the air light at the time, although it is evidently not the identical heat which at the same time affects the thermometer near the surface. It is, however, at the time that the barometer falls that the surface is heated, because solar heat is also then displayed in heating the surface; but it does not then materially disturb atmospheric pressure unless it finds aqueous matter to unite with. When, however, the heat forms an union with water, it passes into the upper regions of the atmosphere, where it acts in expanding the gases, diminishing atmospheric pressure, and producing winds.

In meteorological registers the tension of vapour is sometimes deducted from total atmospheric pressure, and the remainder being treated as dry gaseous matter, is tabulated and occasionally exhibited in curved lines. These lines present an appearance to the eye the reverse of the curve of temperature as shown by the thermometer; and it seems not improbable that this form of exhibiting changes in the atmosphere has contributed

to the retention of the present unsatisfactory mode of treating the subject of meteorology. But it should be borne in mind that the middle of the day is the time at which the sun exerts its power to send vapour into the higher regions, and that at the same time the surface is heated. Yet the heat which is expended in raising the temperature of the surface is evidently not the same that has ascended in vapour to the upper atmosphere; and as the latter portion of heat produces the effects we are tracing, we should endeavour to follow *it*, and discover *its* particular mode of acting. To do this it may be necessary to point out, in the best way we can, in the present state of meteorology, the separate action of the solar heat on dry ground, and on the water which is spread so extensively over the earth.

In places where there is little or no water to be converted into vapour, such as dry sandy deserts, it is well known that the summer sun greatly raises the temperature of the land every day. Even on the sea coast of Lancashire in the summer I have observed that in the middle of the day, while the air over the moist sand of the shore was at 60° the sand itself was at 75°, and the dry sand among the contiguous hills was at 90°. In this case an equal direct supply of solar heat raised the dry sand 15° above that which was moist; but this difference was a consequence of much of the solar heat that reached the ground passing away in vapour from the moist portion. A hygrometer, held in the air at the time, showed that vapour was ascending from the moist shore in great abundance. In dry countries, far from the sea, as less water exists to receive the heat, less vapour is daily raised by the sun, and therefore the temperature of the surface is raised to a higher degree. Burns, when near Bokhara, says: "The heat of the sand rose to 150° and of the atmosphere to 100°;" but then no water was present in the

district around to be converted into vapour. Yet at the same time at more than 15° farther south, in the moist countries of Bengal and Birmah, the temperature of the wet ground was seldom so high as 80° even when the sky was unclouded. Here we see that the solar heat which raised the temperature of dry ground in one part to 150°, raised that of wet ground in warmer latitudes not much more than one-half; but this difference evidently arose from the circumstance that much heat was taken from the wet ground by vapour which passed into the air.

The solar rays make the surfaces of dry deserts, in warm latitudes, very hot, partly because dry air is a bad conductor of heat, and partly because the air does not ascend, as newly-formed vapour does, through another and a colder medium. Ascending currents of warm air are never found over hot deserts. Such parts receive and accumulate heat in the day and part with it during the night without creating material disturbance in the atmosphere. The great cold of the night, as compared with the heat of the day, has been long noticed in the deserts of Arabia, but no such atmospheric disturbances arise from the heat of those parts as are often experienced over certain tropical oceans that have a temperature below 80°, or over the comparatively cool but wet countries of tropical parts. In some deserts the cold of the surface at night is sufficient to condense a part of the little vapour that is in the air into water or dew, but this water is generally reconverted into vapour by the morning sun, without enough being sent into the atmosphere to produce dense cloud and heat the air; therefore, although the barometer might be raised by the morning evaporation, very little mid-day vapour can rise from the already dried ground. In parts of the great African desert where little or no rain falls, much dew is formed on the ground at night by cold, and where the land is level, it is probable that the evaporation of this

dew in the morning produces the appearance of mirage so often spoken of, but the whole of the dew is generally sent into the atmosphere in the form of vapour without creating a dense cloud during the day. In such countries the vapour of the morning may raise the barometer to a certain extent, but if there should not be sufficient condensation of that vapour, in the middle of the day no fall of the barometer can *then* take place. In other and more moist countries enough of water might be evaporated and raised and sufficient vapour be condensed to cause a small decline of the barometer; and in still wetter countries a greater decline might occur, until in such parts as Bombay we have the maximum amount of the daily disturbance.

The Toronto observations were made in a country of lakes, where a large surface of water exists, from which evaporation in the summer takes place freely, and vapour is sent into the air in considerable abundance until rather a late hour of the day; that place therefore is not well calculated to show what occurs over the deserts of Asia and Africa. The central parts of southern Russia offer good sites for examining dry atmospheres, but unfortunately, in the registrations made at the numerous stations of that country, the heights of the wet bulb thermometer are not given in connection with those of the dry one. This omission marks the defective state of meteorology at present, the importance of ascertaining the amount of vapour that is hourly sent into the air by evaporation evidently not being perceived. In Russia, no doubt, as in other parts, it has been supposed that the tension of vapour at all times showed the weight and pressure of all the aqueous matter that was in the atmosphere. But it has been shown that the pressure of vapour, as exhibited by its tension, though itself interesting, is but a part of what is desired to be known, which is, *the causes of the great disturbances in the weight and pressure of the atmosphere in all parts of the world.*

In order to obtain this knowledge it is necessary that the metamorphoses of water should be traced, as has been here attempted in the case of Toronto. We have seen, in that place, that evaporation of water increases atmospheric pressure up to nine o'clock in the morning. But after this hour the vapour parts with its heat of elasticity, *which heat* then adds to the elasticity of the gases, and lightens them, and we wish to know to what extent this is done. The energy, or rate, of evaporation is the best evidence that we have of the quantity of vapour that is sent into the atmosphere, and therefore this energy should as far as possible be shown. In July at Toronto it was at 2°·01 at four o'clock in the morning. At nine it was at 5°·81, and at five in the afternoon it was at 9°·45. Yet after nine in the morning the quantity of vapour found in the air as shown by its tension became less, and it follows that *all the vapour which ascended after nine o'clock, together with the portion that was no longer found in the tension, disappeared;* and there is no doubt that it must have been condensed into liquid in the higher regions. And as we presume that this condensation liberated the heat which caused the fall of the barometer at Toronto, we might infer the extent to which that instrument would sink in other parts, if we could ascertain how much vapour is sent into the atmosphere and condensed under similar circumstances in those parts. Or we may proceed in the reverse course, and from the fall of the barometer infer the amount of vapour that had been condensed in the upper regions.

The aqueous portion exercises so important an influence on the movements of the whole mass of the atmosphere as to require that its metamorphoses should be traced with as much accuracy as possible, and particularly that portion of vapour which ascends from the time that the barometer begins to fall in the morning, until that instrument turns

and rises in the afternoon. To contribute to the attainment of this object I have shown in the table the difference between the temperatures of the wet and of the dry bulb thermometers during the day, which difference expresses the energy of evaporation. *In all meteorological observatories this should be attempted.* The extent to which the wet bulb instrument is kept below the dry one shows the proportional amount of vapour sent into the air. And the hourly heights of each instrument may, during the day, be easily worked on paper ruled to a scale of degrees of temperature. By drawing a straight base line on such paper to represent the wet bulb thermometer, then dotting the height of the dry above the wet one at each hour, and connecting the dots by lines, there would be presented to the eye a curve, showing the relative quantities of vapour that ascend into the atmosphere during each hour. It would be easy to compare these curves of evaporation with others, representing the mid-day falls of the barometer, with a view to tracing their connection. If the same kind of registration were to be made when storms occur in any part of the world, it would almost certainly throw new light on the cause of storms.

The movements of the barometer, as an indicator of atmospheric pressure, are well known, but the influence of vapour, while undergoing condensation into water, on that pressure, is not known, and therefore to this point attention should be strongly directed. As long as the fresh vapour of the morning shows itself only in an increase of tension it may be considered to produce no disturbance of the gaseous atmosphere, though it may raise the barometer. It will, therefore, then be sufficient to register the wet and dry bulb thermometers, the tension of vapour, and the barometer, as these will exhibit what is taking place during the time. But when the barometer begins to fall at nine or ten o'clock the difference between the two thermometers

should not only be carefully tabulated, but graphically exhibited; and if it be found that as more vapour rises the barometer falls, and *to an extent proportioned to the quantity of vapour that is ascending*, it may be taken as evidence that the vapour supplies the means of reducing atmospheric pressure. And if, in addition to this evidence, the tension of the vapour in the lower regions is found to be reduced, that fact may be considered a proof that the quantity of *vapour in the air* has become less because some of the quantity which had previously been there has been condensed; and the barometer alone, by showing how much the atmosphere has been lightened, would be evidence of the total quantity of vapour that had been condensed. In tracing these facts it could hardly fail to be perceived that ascending vapour is the important agent in reducing atmospheric pressure.

From ten o'clock in the morning until four in the afternoon is generally the time when the barometer falls, and during this period the different heights of the wet and dry bulb thermometers show the energy of evaporation—when the tension shows the quantity that remains as vapour,—whilst the fall of the barometer indicates the quantity of heat liberated by condensation of vapour to warm and lighten the atmosphere. Such a system of recording atmospheric changes would present a tolerably clear view of the alterations that take place in the higher regions in the middle of the day, and the facts being presented in a tabular form any meteorologist would be enabled to compare the different phenomena, and to connect them with the working of known laws of nature.

Hourly changes have been traced for the reason already given, viz., that we are better able to collect evidence of their nature than we are respecting those depending on the annual revolutions of the earth. But if we can prove what disturbs the atmosphere at particular hours so much

as to produce a palpable daily change in its weight and pressure, we may apply the knowledge thus obtained to enquiries into the larger alterations that occur irregularly. Little doubt can be entertained that the same cause which makes the barometer decline daily, reduces atmospheric pressure during storms to a larger extent. The daily reduction of pressure, when no rain falls, is not often more than equal to half an inch of mercury; but if condensation has begun, at any time when there is a sufficient supply of vapour, it may go on until the atmosphere becomes much heated in one part to a great height, and its weight in that part may be so much reduced, that the barometer may fall one or two inches — or even more, — the vapour to effect this large decline being brought from distant parts by winds.

V. — *On the general solution of the problem of the polyedra. By the* Rev. Tho. P. Kirkman, M.A., F.R.S., *Rector of Croft with Southworth, and Honorary Member of the Literary and Philosophical Society of Manchester.*

Read January 26th, 1858.

Long as the question — How many x-edra are there? — has been before the mathematical world, no method has hitherto been discovered, except that of tentative construction, whereby the number required can be completely assigned even for $x=8$. All that has been effected, so far as I can learn, towards the solution of the problem by general methods, may be seen in four memoirs of mine printed in the *Philosophical Transactions for* 1856 *and* 1857, and in a later memoir "On the partitions of the r-pyramid," of which some account is given in the *Proceedings of the Royal Society for* 1857,* and in another "On the triedral partitions of the r-ace" in the *Manchester Memoirs for* 1857. In these papers are discussed certain great classes of the x-edra. In the pages following is made the first attempt at the analysis and solution of the whole problem.

It is known that if the Q′ faces of p'-acral Q′-edron be an A-gon, a B-gon $\cdots$ a Q-gon, and its p' summits an a-

* This memoir is now printed in the *Philosophical Transactions.*

ace, a b-ace $\cdots$ a p-ace, e being the number of its edges, the equation following will be satisfied.

$$(A)^* \quad A+B+C+\cdots+Q=a+b+c+\cdots+p$$
$$=2e=2(p'-1+Q'-1);$$

and there are in general many p'-acral Q'-edra, having a common description in this equation. But to determine whether this equation, when values of AB, &c. are chosen, really describes polyedra, and how many it includes, all differing in the mutual arrangement of the faces and summits specified, has hitherto been found a problem of insuperable difficulty. This difficulty arises from the fact that the problem is chiefly *tactical*; and we have had no *tactic calculus*, nor any means of expressing tactical conditions in algebraic language. It is easy enough to multiply theorems in combination, when these are symmetrical and exhaustive; but to assign the number of combinations or systems of combinations that can be selected, so as to fulfil given unsymmetrical conditions, and to be all essentially distinct, has thus far been a task beyond the powers of our algebra.

The question, How many x-edra are there? is not without interest for its own sake; for every x-edron is one out of a precise number of permanent facts in nature: its principal charm, however, has lain in its difficulty, and in the consideration that the answer has been part of the sublime unknown. It is precisely on problems like this that the modern analysis delights to try the edge and temper of her tools; or rather to exercise herself in the handling of instruments which, though lent to human workmen, are not of human workmanship, but whose fabric and finish are divine.

The obstacle to research in this direction has been

* Vide my paper "On the representation and enumeration of the Polyedra." — *Manchester Memoirs*, vol. xii.

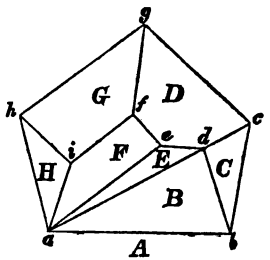
mainly the difficulty of stating the conditions of the problem in algebraical propositions, *i.e.* in equations. In this paper the reader will find that this statement is effected, and that nothing remains to be done for the complete solution except a certain amount of laborious elimination.

The first step is to invent a representation of a polyedron on paper in symbols that algebra can handle. The consideration of the solids in space will help us little. I have tried to refer the figure to its amplest, *i.e.* most angled, face as a base, and the figure has obstinately refused to acknowledge one face rather than another for a foundation. Its amplest face may be a q-gon, but it may have a hundred of these q-gons, about every one of which may stand a different configuration of summits and faces. I have endeavoured to generate the figures by defined processes from pyramids; but not only is the solid in general deducible from different pyramids, but often in a great number of ways from the same pyramid, no one of which will allow another to be a better way than itself. And when I have attempted by authority to put the work into the hands of some one of these methods, all equally clamorous and confident, I have invariably found that, in spite of all I could do to prevent it, the operator would persist in carving the same polyedron over and over again in different postures, so that it was impossible to know how many really distinct ones had been generated.

The true method of looking at the question turns out, as is usually the case, to the mortification of wasted ingenuity, to be a very simple one.

All that we require is to have a clear statement of the summits that are about every face, and of the faces about every summit, and of the order about each in which they stand. This order is fixed by a correct account of the

edges, showing the two faces and the two summits bounding each edge; for if the edges E and F make an angle on the solid, they will have then both a common face and a common summit.



Thus to form a paradigm of the 9-acral 8-edron here figured, which is one of those described by the equation,

$$5_A+5_D+4_F+4_G+3_B+3_C+3_E+3_H$$
$$=5_a+4_d+3_b+3_c+3_e+3_f+3_g+3_h+3_i=2\cdot 15,$$

we mark the faces in any way A B C $\cdots$, and the summits $a\ b\ c\cdots$ We next form a table of eight columns, headed A B C $\cdots$, making nine rows headed by $a\ b\ c\cdots$,

		5	5	4	4	3	3	3	3
(A)	5	A_a	·	F_a	·	B_a	·	E_a	H_a
	4	·	D_d	·	·	B_d	C_d	E_d	·
	3	A_b	·	·	·	B_b	C_b	·	·
	3	A_c	D_c	·	·	·	C_c	·	·
	3	·	D_e	F_e	·	·	·	E_e	·
	3	·	D_f	F_f	G_f	·	·	·	·
	3	A_g	D_g	·	G_g	·	·	·	·
	3	A_h	·	·	G_h	·	·	·	H_h
	3	·	·	F_i	G_i	·	·	·	H_i

putting M_n for the point common to the column M and the row n, and entering $M_n=1$, or $M_n=0$, according as the face marked M has or has not the summit marked n for one of those about it, *i.e.* according as the summit has or has not the face for one of those about it.

Thus we have entered $A_a=1=F_a=B_a$, because the pentace a is in the pentagon A, the tetragon F and the triangle B; and $D_a=0=G_a$, because that pentace is not

in the pentagon D, nor in the tetragon G. Hence it comes to pass that we read five units under the pentagon A, showing all its summits, and five units opposite the pentace a, showing all its faces, &c.

The fifteen edges of the figure are represented thus:
$A_aA_bB_aB_b=1$, $A_bA_cC_bC_c=1$, $A_cA_gD_cD_g=1$, $A_gA_kG_gG_k=1$, $A_kA_aH_kH_a=1$, $D_dD_eE_dE_e=1$, $B_bB_dC_bC_d=1$, $B_aB_dE_aE_d=1$, $C_cC_dD_cD_d=1$, $D_fD_gG_fG_g=1$, $D_eD_fF_eF_f=1$, $G_fG_iF_fF_i=1$, $G_kG_iH_kH_i=1$, $F_aF_iH_aH_i=1$, $F_aF_eE_aE_e=1$.

Each factor in every quadruplet is unity. The paradigm shows these edges each one as four points of a rectangle; thus A_c A_g D_c D_g are four such points, and F_a F_i H_a H_i are such also. The charming peculiarity of the paradigm is, that no displacement of any face and its column, or of any summit and its row, can break these edges, *i.e.* spoil these rectangles; so that the same polyedron is accurately exhibited, whatever be the order of the letters A B C$\cdots$ or $a\ b\ c\cdots$ And no rectangle occurs in the paradigm which is not an edge of the polyedron. The property of these quadruplets is, that no duad of two capitals or of two small letters appears a second time, while every duad of a capital and a small letter occurs exactly twice over, and always in two different quadruplets.

If the reader will take any other 8 numbers, A B C$\cdots$ and some 9 $a\ b\ c\cdots$, satisfying such an equation as (A), and endeavour by trial to form with them a paradigm showing fifteen edges, he will, whether he succeeds or not, obtain some idea of the tactical difficulty involved in this problem of the polyedron. Or he may try to satisfy himself as to the possibility of forming other 9-acral 8-edra with the numbers above handled.

The next step is to express all the conditions of the question in the shape of equations, which are to be true independently of all arrangement of their symbols.

We have $p'Q'$ unknown quantities M_n, discontinuous variables, which can have any of them the value zero or unity, and no other value. The columns headed by A B C $\cdots$, the faces in (A), give the equations,

$$(M_n)\quad \begin{aligned} A&=A_a+A_b+A_c+\cdots+A_p=A_a^r+A_b^r+\cdots+A_p^r,\\ B&=B_a+B_b+B_c+\cdots+B_p=B_a^r+B_b^r+\cdots+B_p^r,\\ &\vdots\quad\vdots\quad\vdots\quad\vdots\qquad\vdots\quad\vdots\quad\vdots\qquad\vdots\\ Q&=Q_a+Q_b+Q_c+\cdots+Q_p=Q_a^r+Q_b^r+\cdots+Q_p^r, \end{aligned}$$

Q' equations for any whole value >0 of r, because $M_n^r = M_n$, whether this be unity or zero, and the rows give the p' equations for any value integer and positive of r.

$$(M_n)\quad \begin{aligned} a&=A_a+B_a+C_a+\cdots+Q_a=A_a^r+B_a^r+\cdots+Q_a^r,\\ b&=A_b+B_b+C_b+\cdots+Q_b=A_b^r+B_b^r+\cdots+Q_b^r,\\ &\vdots\quad\vdots\quad\vdots\quad\vdots\qquad\vdots\quad\vdots\quad\vdots\qquad\vdots\\ p&=A_p+B_p+C_p+\cdots+Q_p=A_p^r+B_p^r+\cdots+Q_p^r. \end{aligned}$$

The $(p'-1+Q'-1)$-edges have next to be expressed. If one of these is the intersection of the faces I J, and connects the summits m and n, we shall have

$$I_mI_nJ_mJ_n=1, \text{ and}$$
$$I_mI_nK_mK_n=0=I_mI_pJ_mJ_p,$$

whatever be K and p; because the three faces I J K, having no common line, cannot all contain the same two summits m and n, and the three summits $m\,n\,p$, not being in the same line, cannot all be in the same two planes I and J.

As the I-gonal face I has I edges, we have the following equations, by what has just preceded,

$$(E)\quad \begin{aligned} &\Sigma(A_mA_nB_mB_n+A_mA_nC_mC_n+A_mA_nD_mD_n+\cdots\\ &\qquad+A_mA_nQ_mQ_n)=A\\ &\Sigma(B_mB_nA_mA_n+B_mB_nC_mC_n+B_mB_nD_mD_n+\cdots\\ &\qquad+B_mB_nQ_mQ_n)=B\\ &\Sigma(C_mC_nA_mA_n+C_mC_nB_mB_n+C_mC_nD_mD_n+\cdots\\ &\qquad+C_mC_nQ_mQ_n)=C\\ &\qquad\vdots\qquad\qquad\vdots\\ &\Sigma(Q_mQ_nA_mA_n+Q_mQ_nB_mB_n+Q_mQ_nD_mD_n+\cdots \end{aligned}$$

$$+Q_mQ_nP_mP_n)=Q,$$

or more briefly,

$$\Sigma I_mI_nX_mX_n=1,$$

where X is every letter in turn but I out of A B C$\cdots$Q, and m n is *every pair in turn* of a b $c\cdots p$.

In like manner, since the m-edral summit m has m edges, we must have

$$\begin{aligned}(E)\quad &\Sigma(V_aX_aV_bX_b+V_aX_aV_cX_c+V_aX_aV_dX_d+\cdots\\ &\quad+V_aX_aV_pX_p)=a\\ &\Sigma(V_bX_bV_aX_a+V_bX_bV_cX_c+V_bX_bV_dX_d+\cdots\\ &\quad+V_bX_bV_pX_p)=b\\ &\quad\vdots\\ &\Sigma(V_pX_pV_aX_a+V_pX_pV_bX_b+V_pX_pV_cX_c+\cdots\\ &\quad+V_pX_pV_oX_o)=p,\end{aligned}$$

or more briefly,

$$\Sigma V_mX_mV_xX_x=m,$$

where x is every one in turn but m of a b $c\cdots p$, and V X is *every pair in turn* of A B C$\cdots$Q.

We have thus completely expressed both the tactical and arithmetical conditions of the problem.

I fear that there is no shorter mode of arriving at the solution than by the way of elimination of this table-full of $p'Q'$ variables. This is possible; but the enormity of the operations implied is such as to give a very lively idea of the difficulty and vastness of this problem of the polyedra.

The $p'+Q'$ equations $(M_n)(r=1)$ give the means of eliminating as many variables from the $p'+Q'$ equations (E). The values thus found for those $p'+Q'$ are then to be substituted in the equations $(M_n),(r=2)$,

$$A=A_a^2+A_b^2\ \&c.,\quad B=B_a^2+B_b^2\ \&c.,$$
$$a=A_a^2+B_a^2\ \&c.,\quad b=A_b^2+B_b^2\ \&c.,$$

which will introduce certain duad products; after which all exponents may be erased from the variables, and $p'+Q'$

more can be eliminated as before from equations (E). The values substituted for the expelled variables are next to be introduced in

$$A = A_a^3 + A_b^3 \text{ &c.}, \quad B = B_a^3 + B_b^3 \text{ &c.}, \text{ &c.},$$
$$a = A_a^3 + B_a^3 \text{ &c.}, \quad b = A_b^3 + B_b^3 \text{ &c.};$$

by which operation certain products of three, four, five and six variables will be exhibited in the last written $p' + Q'$ equations. We can then erase exponents and proceed linearly to expel $p' + Q'$ more variables from equations (E), and can thus operate till all are eliminated; when an equation $U = 0$ will be the result, expressing a condition or conditions among the quantities $A\,B \cdots Q$, $a\,b \cdots p$, which will be different from (A).

The work will simplify itself as we proceed, by the erasure of all those products of variables which contain the capital J with more than J subindices, or the subindex i under more than i capitals; for we know that not more than J of $J_a\,J_b\,J_c$, &c., can have value, nor more than i of $A_i\,B_i\,C_i$, &c. Or, if we wish to obtain a result in terms perfectly general, without any choice of value for J or i, we can content ourselves with erasing all exponents. Of course no product of more than Q' *different* capitals, or more than p' different subindices, can possibly remain.

If we wish to construct and enumerate the polyedra of which (A) is the common description, we may do it thus. First eliminate all capitals but A. The mere inspection of the result in $A_a\,A_b\,A_c$, &c., will show what systems of A values are possible, for $A_m = 0$ $A_m = 1$ are the only values that can arise, that is, no system of summits can be about the A-gon, which is not *readable* in that result; and if there be only one A-gon, it will often happen that only one set of summits is readable and admissible. We can pronounce with certainty that all are impossible but those (m n, &c.) readable in our result ($A_m = 1$ $A_n = 1$, &c.)

We next eliminate all capitals but A and B, obtaining an equation containing only $A_a A_b \cdots B_a B_b \cdots$. Putting in this for A_a, &c., in turn the systems of A values not proved before impossible, we can read a certain number of sets of B values ($B_m = 1$ $B_p = 1$, &c.) of which we can say that none besides are possible; that is, we can construct a certain number of sets of two columns under A and B, of which we can say that every possible system is written down among them. Next, forming the equation containing only $A_a A_b \cdots B_a B_b \cdots C_a C_b \cdots$, we can construct a limited number of sets of three columns, of which we can say with certainty that every possible polyedron is thus far constructed. And thus we can proceed till we have before us a certain number of sets of Q′ columns, of which we can pronounce that every possible Q′-edron is once, and only once, constructed; that is, we can, by a strictly demonstrative process, both construct and enumerate the p'-acral Q′-edra described in equation (A). And this is a complete and rigorous solution, from the theoretical point of view, of the problem of the polyedra; for they are all comprised in descriptions of the form of (A).

The result $U = 0_1$ above mentioned, will show at once whether any polyedron is described in equation (A); for, if none exists, $U = 0$ will not be satisfied by the values of $A B \cdots Q\ a b \cdots p$ under consideration.

It appears highly probable that the same equation $U = 0$ will be sufficient for the enumeration of the different polyedra under (A), for the reasons following.

Let $\Sigma X_r X_a Y_a Y_s$ be the number of pairs of lines, edges or diagonals, drawn or drawable from the a-ace a to any two other summits r and s, $a r$ being in the X-gon and $a s$ in the Y-gon, which have the common summit s. No two different polyedra can have this sum alike at every summit a; for if they have, they will have the same faces in the

same order about every summit, *i.e.* they will not be different polyedra. It is easy to form equations containing such sums. Thus, if we multiply the square of A by that of B, we have

$$A^2=\Sigma A_r^2+2\Sigma A_rA_s=\Sigma A_r+2\Sigma A_rA_s=A+2\Sigma A_rA_s$$
$$B^2=\Sigma B_r^2+2\Sigma B_rB_s=\Sigma B_r+2\Sigma B_rB_s=B+2\Sigma B_rB_s,$$
$$\tfrac{1}{2}(A^2-A)(B^2-B)=\Sigma A_rA_s\Sigma B_rB_s,$$

where $r\,s$ is every pair in turn of $a\,b\cdots p$, in each of the sums of this product. Hence, keeping A invariable, and giving to X $r\,s$ every possible value,

$$\Sigma\tfrac{1}{2}(A^2-A)(X^2-X=\Sigma A_rA_sX_rX_s+\Sigma A_rA_sX_sX_t+\Sigma A_rA_sX_tX_u$$
$$=A+\Sigma A_rA_sX_sX_t+\Sigma A_rA_sX_tX_u;$$

which, by adding up Q′ such equations, one for every face in turn of A B $\cdots$ Q, becomes

$$\tfrac{1}{2}\Sigma(X^2-X)(Y^2-Y)=2e+2\Sigma X_rX_sY_sY_t+2\Sigma X_rX_sY_tY_u,$$

where X and Y are every pair in turn of A B $\cdots$ Q, and $r\,s\,t\,u$ are every set in turn of four out of $a\,b\cdots p$. In like manner it is proved that

$$\tfrac{1}{2}\Sigma(r^2-r)(s^2-s)=e+\Sigma X_rY_rY_sZ_s+\Sigma X_rY_rV_sZ_s$$

where $r\,s$ are every pair in turn of $a\,b\cdots p$, and X Y V Z are every four in turn out of A B $\cdots$ Q.

So far as I have examined cases, I find no two different polyedra having the same value of the sum $\Sigma X_rX_sY_sY_t$ taken over all the summits s.

Now there will be for every polyedron a certain equation $S=0$, containing only symmetric functions of the variables of different degrees in them, and true for no other polyedron. This will be translateable into an equation $V=0$, containing only symmetric functions of the numbers A B $\cdots$ Q $a\,b\cdots p$. This condition $V=0$ will therefore, in all probability, be a factor of our result of elimination, $U=0$. If so, the number of factors of distinct forms (such that the evanescence of one does not for all values of A B, &c., follow from that of the other), into which $U=0$ can be

broken, and which are satisfied by the values of A B, &c., under review, will be the number of p'-acral Q′-edra described under equation (A).

There is reason to believe that the result of this investigation may be made to take the shape

$$H = Ix + Jy + Kz + \cdots$$

where H I J K $\cdots$ are symmetric functions of A B $\cdots$ Q $a\,b\cdots p$, and $x\ y\ z\cdots$ are symmetric functions of the variables M_n; and such a shape that the number of solutions of this equation in whole values of $x\ y\ z\cdots$ is the exact number of polyedra which it is our object to discover. This form, if it be attainable, would certainly be the most elegant issue that we could desire to our investigation.

As the number of these symmetric functions is limited, and as they appear in the expression of symmetric functions of the numbers A B C $\cdots$ $a\ b\ c\cdots$, it is plain that all but certain ones can be eliminated; but I have not been able to obtain a final equation having *no solutions* in $x\ y\ z$ $\cdots$ *foreign* to the question. The equation above found,

$$\tfrac{1}{4}\Sigma(X^2-X)(Y^2-Y) - e = x + y$$

is true for more values of x and y than those of which we are in quest. But as I have not formed the equation obtained by eliminating from all the equations of this kind that can be written, I cannot say whether a result, showing symmetric functions of higher degrees, can be obtained of the form desired.

I have a demonstration that the number of p'-acral Q′-edra can be easily found, whatever be the numbers of A B $\cdots$ Q $a\,b\cdots p$, if that of the p'-acral Q′-edra be known for another set of numbers $A_1\ B_1\cdots Q_1\ a_1\ b_1\cdots p_1$, of which either $A_1 = B_1 = C_1 = \cdots = Q_1 = 3$, or else $a_1 = b_1 = c_1 = \cdots = p_1 = 3$. And thus the problem is reduced to the determination of the number of Q-edra having only triedral summits, or, which is analytically the same thing, of p-

acra having only triangular faces. But I do not see any way to the solution of the problem, even in this shape, more direct than that given in the pages preceding. In fact, the question of the x-edra having only triedral summits is identical with this *problem in Tactics*:

To form triplets of three different letters with A A's, B B's, C C's, *&c., so that* all *these symbols shall be once, and once only, employed, and that every possible duad,* A B, A C, *&c., shall be either not or twice employed.*

The analysis above given supplies a complete solution hereof; and this is, I believe, the first instance in which a problem of this nature has found either an entire solution, or even a translation into algebraical language. I have my hopes that the method here opened will suggest the proper expression and solution of a multitude of tactical problems, which have hitherto bidden defiance to our analysis.

VI. — *Experiments to Determine the Properties of some Mixtures of Cast Iron and Nickel.*

By WILLIAM FAIRBAIRN, F.R.S.

Read March 9th, 1858.

SOME of my chemical friends in London had got an impression from a careful analysis of meteoric iron, that it could be produced artificially by the combination of some of the same elements that were found to exist in the specimen analysed, containing about 2½ per cent of nickel.

In order to determine whether it would be possible to obtain an artificial compound of this nature, and to ascertain the effect produced by mixing a certain proportion of nickel with cast iron, the following experiments were instituted. They consisted, in the first instance, in the extraction of the nickel from the ore which is found in the mines of the Duke of Argyle, near Inverary in Scotland. The metal having then been purified by repeated meltings, was mixed with cast iron in such proportions as to form a compound, containing the same quantity of nickel as the specimen analysed. This was done by melting 2½ per cent of nickel with carefully selected South Welsh cast iron from the works of Blaenarvon and Pontypool. The mixtures were fused in crucibles, and run into ingots or bars, which were then tested in regard to their mechanical powers of resistance to a transverse strain.

This was done with great care, and the results which follow give unmistakeable evidence of the effects produced upon cast iron by an admixture of nickel, however small the quantity of the latter that may be introduced. Meteoric iron is, above all others, the most ductile, and it is recorded by travellers that the Esquimaux have instruments made from this description of iron so ductile that they may be made to bend round the arm. The ingots prepared on the occasion of these experiments were, however, widely different, as their power to resist impact was nearly one half less than in those composed of pure iron.

It is uncertain what might have been the results had the castings produced been treated as cast steel, and hammered out until they were rendered malleable and magnetic. This process was not, however, attempted, as, judging from the appearance of the fracture, they were more likely to crumble under the hammer than attain malleability.

The nickel for these experiments was prepared from the ore, by fusing at a very high temperature in a crucible or steel pot,

30*lbs.* of roasted ore,
5*lbs.* of fine sand,
2*lbs.* of charcoal,
2*lbs.* of lime.

This mixture was kept in the furnace six hours, and then taken out and allowed to cool. The metal was then separated from the slag, and again melted with half its weight of roasted ore and one quarter its weight of green bottle glass ground to powder.

As before, the mixture was kept for six hours, at the temperature of a cast steel furnace. The metal had by the end of that time collected at the bottom of the crucible. It contained about 25 per cent of nickel, and was

of sufficient purity to be fused with cast iron. 10*lbs.* of it melted with 112*lbs.* of cast iron gave a mixture containing about 2½ per cent of nickel.

The object of these fusings was to reduce the metallic oxide by means of the charcoal, while the lime and sand removed the oxide of iron, silica, sulphur and other impurities by forming a fusible slag.

Mixtures of nickel with cast iron (No. 1) and Blaenarvon iron (No. 3) and Pontypool iron, each containing about 2½ per cent of nickel, and also some pure Blaenarvon cast iron (No. 1) were cast into bars about one inch square and two feet six inches long, and subjected to the following experimental tests: —

TABLE I. Breaking weights and deflections of bars of cast iron and of alloys of nickel and cast iron, when subjected to a transverse strain.

Experiment I. Cast Iron No. 1, with Nickel.		Experiment II. Pure Cast Iron, No. 1.		Experiment III. Blaenarvon Iron, No. 3, with Nickel.		Experiment IV. Blaenarvon Iron, No. 3.		Experiment V. Pontypool Iron with pure Nickel.	
Section of bar in inches, 1×1.		Section of bar in inches, 1×1.		Section of bar in inches, 1×1.		Section of bar in inches, 1.02×1.07.		Section of bar in inches, 1×1.	
Weight laid on in lbs.	Deflection in inches.	Weight laid on in lbs.	Deflection in inches.	Weight laid on in lbs.	Deflection in inches.	Weight laid on in lbs.	Deflection in inches.	Weight laid on in lbs.	Deflection in inches.
147	0·125	147	0·1	147	0·05	147	0·05		
259	0·175	259	0·155	259	0·1	259	0·1		
315	0·19			315	0·14				
371	0·25	371	0·2	371	0·16	371	0·115	371	0·225
427	0·275			427	6·18	427		427	0·237
483	0·34	483	0·225	483	0·2	483	0·2	483	0·25
539	0·375	539	0·25	539	0·237	539	0·237	539	0·262
				567	0·25	567	0·25		
595	0·41	595	0·31	595	0·255	595	0·262		
		609	0·325						
623	0·425	623	0·33	623	0·277	623	0·275		
637		637	0·35						
Broke with this weight.		651	0·36	651	0·285	651	0·3	651	0·3
		679	0·365	679	0·29	679	0·325	679	0·312
		707	0·367	707	0·3	707	0·337	707	0·325
		721	0·375						
		735	0·4	749	0·325	749	0·362		
		763	0·41	763	0·34			763	0·35
		777	0·425	791	0·425	777	0·375	791	
		791	0·44						
		819	0·46	833	0·45	819	0·4	798	
		847	0·46			847	0·42	Broke with this weight.	
		861		861					
		Broke with this weight.		875		875	0·437		
				Broke with this weight.		917	0·45		
						959	0·5		
						1043	0·525		
						1099	0·575		
						1155	0·675		
						With this weight it remained twenty minutes.			
						1267	0·75		
						Broke after some seconds.			
Ultimate deflection 0·434″		Ultimate deflection 0·47″		Ultimate deflection 0·58″		Ultimate deflection 0·75″		Ultimate deflection 0·366″	

TABLE II. Results reduced to bars 1″ *square. Distance between the supports* 2′ 3″.

	Breaking weight (b)	Ultimate deflection (d)	Power of resisting impact ($b \times d$)	Strength compared with Blenarvon =1000
Experiment I. bar D. pure Blaenarvon No. 3.....	1131	0·75	848·2	1000
Do. II. „ C. Blaenarvon No. 3 and nickel...	875	0·58	507·5	773
Do. III. „ B. pure cast iron No. 1.....	861	0·47	404·7	761
Do. IV. „ A. cast iron No. 1 and nickel.....	637	0·434	276·4	563
Do. V. „ E. Pontypool iron and pure nickel	798	0·366	292·1	705
Mean.....	860	0·52	465·7	760

From the above it is evident that an admixture of nickel in the proportion of 2½ per cent does not increase but diminish the tenacity of cast iron. To what extent it might be improved by augmenting or lessening the proportion of nickel, a more extended series of experiments alone can determine. Mixtures of the two metals in the proportion used in the above experiments are decidedly inferior to the pure metal in their power of resistance to a transverse strain and to impact. In the first and second experiments on Blaenarvon iron there is a loss of nearly one-fifth the strength; or the strength of the pure metal is to that of the mixture as 1000 : 773. And in Experiments III. and IV. with a more fluid iron there is about the same loss, the relative strengths being as 761 : 563; or as 1000 : 740. From these facts it is evident that nickel in the proportion of 2½ per cent seriously injures the strength of cast iron, and moreover has injurious effects on its power of resisting impact, as the columns in the above table indicating those properties clearly show.

It is difficult to account for the serious deterioration and loss of strength which the above experiments indicate. It may probably arise from the improper treatment of the nickel ore during its calcination and subsequent reduction

in the crucible. When cast into ingots after the second melting the nickel had not the appearance of a pure metal, but exhibited a dull fracture, as if fine sand or particles of earth had been mixed with the crystals, showing the presence of impurities which it would be almost impossible to get rid of. It remained, therefore, a question for consideration whether the results would be different if nickel, properly prepared and of greater purity, were employed. To clear up doubts on this point a mixture of pure nickel with No. 3 Pontypool iron was made, but the result was a bar of white silvery metal, which broke when the weight of 798*lbs.* was laid on, as will be seen by referring to the Tables above.

I obtained from London a number of other bars, consisting of iron and nickel, which on being submitted to the same tests as before, gave better results than those obtained in the previous experiments when nickel prepared from the ore was employed.

The results obtained from this second series of bars are given in the following table:

TABLE III. Breaking weights and deflections of bars of nickel and cast iron, when subjected to a transverse strain. The bars were about one inch square, and the distance between the supports was two feet three inches.

Experiment VI. Bar F 1 not marked with notches.		Experiment VII. Bar F 2 not marked with notches.		Experiment VIII. Bar G 1 marked with one notch.		Experiment IX. Bar G 2 marked with one notch.		Experiment X. Bar H 1 marked with two notches.		Experiment XI. Bar H 2 marked with two notches.	
Section of bar in inches, 1·02 × 1·02.		Section of bar in inches, 1·02 × 1·02.		Section of bar in inches, 1·02 × 1·02.		Section of bar in inches, 1·2 × 1·2.		Section of bar in inches, 1·15 × 1·15.		Section of bar in inches, 1·15 × 1·15.	
Wght. laid on in lbs.	Deflection in inches.	Wght. laid on in lbs.	Deflection in inches.	Wght. laid on in lbs.	Deflection in inches.	Wght. laid on in lbs.	Deflection in inches.	Wght. laid on in lbs.	Deflection in inches.	Wght. laid on in lbs.	Deflection in inches.
147	0·02	147	0·005	147	0·02						
259	0·06	259	0·05	259	0·06	259	0·06	259	0·06	259	0·06
371	0·1	371	0·1	371	0·12	371	0·08	371	0·1		
427	0·12	427	0·115	427							
483	0·14	483	0·14	483	0·17	483	0·11	483	0·14	483	0·14
539	0·16	539	0·18	539		539	0·155	539	0·165		
595	0·18	595	0·19	595	0·23	595	0·17	595	0·17	595	0·18
623	0·19										
651	0·2	651	0·21	651	0·26	651	0·19	651	0·18	651	0·2
679	0·21	679	0·22								
707	0·22	707	0·23	707	0·29	707	0·2	707	0·19		
735	0·24	735	0·24								
763	0·25	763	0·255	763	0·32	763	0·215	763	0·2	763	0·23
777	0·255										
791	0·26	791	0·27	791							
805	0·265	805	0·28	Broke.		819	0·23	819	0·23	819	0·25
819	0·27	819	0·285			875	0·25	875	0·24	875	0·27
833	0·28	833	0·29			931	0·27	931	0·27	931	
847	0·29	847	8·295			959	0·28	959	0·28	Broke.	
861	0·3	861	0·3			987	0·285	987			
875	0·31	875	0·31			1043	0·3	Broke.			
903		903	0·32			1057	0·31				
Broke.		917	0·325			1071	0·317				
		931	0·33			1099	0·321				
		959	0·345			1127	0·33				
		987	0·355			1155	0·34				
		1015	0·375			1183	0·35				
		1029	0·38			1121	0·36				
		Broke.				1239	0·375				
						1267	0·385				
						1295					
						Broke.					
Ultimate deflection 0·315″		Ultimate deflection 0·38″		Ultimate deflection 0·331″		Ultimate deflection 0·41″		Ultimate deflection 0·286″		Ultimate deflection 0·29″	

TABLE IV. Results reduced to bars 1″×1″. *Distance between the supports* 2′ 3″.

	Breaking weight (b)	Ultimate deflection (d)	Power of resisting impact. ($b \times d$)	Rates of strength Bar F 2 =1000
Experiment VI. Bar F 1 without notches	867	0·315	273	1000 : 876
Do. VII. „ F 2 „ „	989	0·38	376	1000 : 1000
Do. VIII. „ G 1 with one notch.	760	0·331	231	1000 : 768
Do. IX. „ G 2 „ „	899	0·41	368	1000 : 908
Do. X. „ H 1 with two notches	746	0·286	213	1000 : 754
Do. XI. „ H 2 „ „	703	0·29	203	1000 : 810
Mean.....	829	0·335	280	1000 : 838

I have been unable to ascertain the precise composition of these bars, but assuming it to have been similar to that of the first series of bars, the greater powers of resistance shown by them would seem to indicate that the nickel employed in their preparation possessed a higher degree of purity than that used for the first series. Much, however, depends on the quality of the cast iron with which the nickel is mixed. The results derived from the foregoing experiments are conclusive, both in regard to those made on the first and those made on the second series of bars. Further experiments may, however, lead to different results; but judging from what has already been done, I am inclined to believe that chemical combinations of a different nature are required, and probably a totally different process of manufacture will have to be adopted before a sufficiently strong and satisfactory compound can be obtained.

In attempting to ascertain the effect of a mixture of nickel with cast iron, the principal object was to determine to what extent the compound gave positive or negative results. It is well known that meteoric iron is peculiarly ductile, and it was assumed that nickel, added to cast iron in such proportion as to produce a compound similar

to meteoric iron, would impart to it increased ductility. The foregoing experiments lead, however, to the conclusion that an admixture of nickel produces an exactly opposite effect, and it now remains to be determined by a more extended series of experiments, whether by mixing nickel with malleable iron in the same relative proportion more satisfactory results would be obtained. In prosecuting these experiments it would be interesting to know the extent to which these metals are capable of combining chemically, and how closely such combinations would approximate to meteoric iron.

Besides endeavouring to obtain a metal of greater ductility, another object of equal importance was aimed at in these experiments, namely, to produce a metal of increased tenacity suitable for the casting of cannon and heavy ordnance. During the last two years innumerable experiments have been made for this purpose, with more or less success; but the ultimate result appears to be, that for the construction of heavy artillery there is no metal so well calculated to resist the explosion of gunpowder, as a perfectly homogeneous mass of *the best and purest cast iron*, when freed from sulphur and phosphorus.

VII. — *On the Hardness of Metals and Alloys.*

By F. Crace Calvert, M.R.A. *of Turin*, F.C.S. *&c.*; *and* Richard Johnson, F.C.S., *&c.*

Read April 6th, 1858.

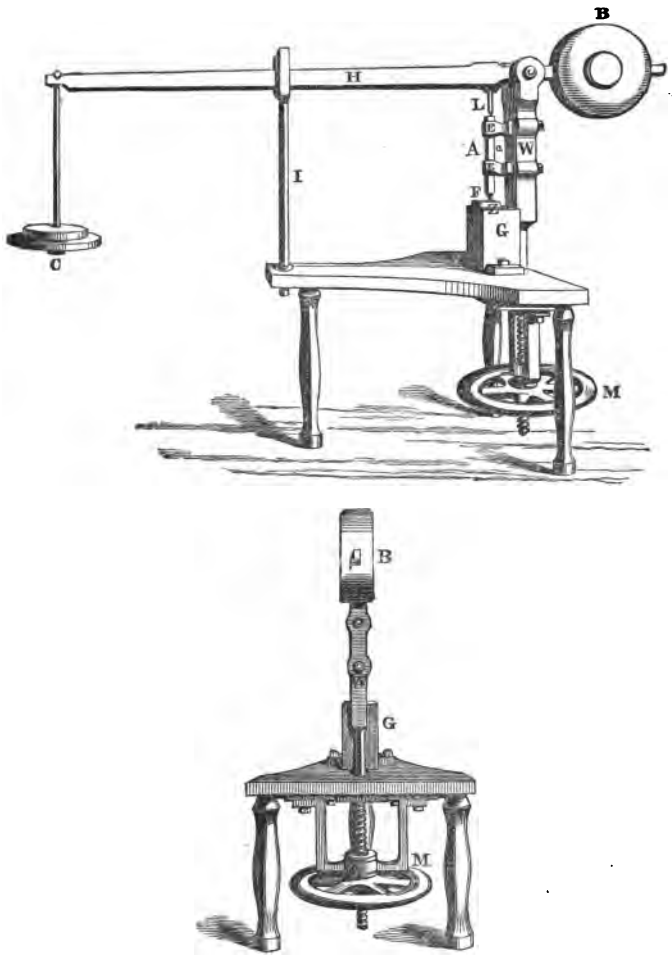
The process at present adopted for determining the comparative degree of hardness of bodies consists in rubbing one body against another, and that which indents or scratches the other is admitted to be the harder of the two bodies experimented upon. Thus, for example,

Diamond,	Iron,
Topaz,	Copper,
Quartz,	Tin,
Steel,	Lead.

This method is not only very unsatisfactory in its results, but it is also inapplicable for determining with precision the various degrees of hardness of the different metals and their alloys. We therefore thought that it would be useful and interesting if we were to adopt a process which would enable us to represent by numbers the comparative degrees of hardness of various metals and their alloys.

To carry out these views we devised the following apparatus and method of operating. The machine used is on the principle of a lever, with this important modification, that the piece of metal experimented upon can be relieved from the pressure of the weight employed without removing the weight from the end of the longer arm of the

lever. The machine consists of a lever H with a counterpoise B and a plate C, on which the weights are gradually placed. The fulcrum L bears on a square bar of iron A,



passing through supports E. The bar A is graduated at *a*, and has at its end a conical steel point F, 7*mm.* or 0·275 of an inch long, 5*mm.* or 0·197 of an inch wide at the base, and 1·25*mm.* or 0·049 of an inch wide at the point which

bears on the piece of metal Z to be experimented on, and this is supported on a solid piece of iron G. The support or point of resistance W is lowered or raised by the screw M, and when, therefore, this screw is turned the whole of the weight on the lever is borne by the support I and the screw M. When it is necessary, by turning the screw M, the weight on the lever is re-established on the bar, and experimented upon.

When we wished to determine the degree of hardness of a substance we placed it on the plate G, and rested the point F upon it, noticing the exact mark on *a* on the bar A, and then gradually added weights on the end of the lever C until the steel point F entered 3·5*mm.* or 0·128 of an inch during half an hour, and then read off the weight. A result was never accepted without at least two experiments were made, which corresponded so far as to present a difference of only a few pounds. The following table gives the relative degree of hardness of some of the more common metals. We specially confined our researches to this class, wishing the results to be practically useful to engineers and others who have to employ metals, and often require to know the comparative hardness of metals and alloys.

Names of Metals.	Weight employed.	Calculated Cast Iron = 1000.
	lbs.	
Staffordshire Cold Blast Cast Iron — Grey, No. 3	4800	1000
Steel	4600 ?	958 ?
Wrought Iron*	4550	948
Platinum	1800	375
Copper — pure	1445	301
Aluminium	1300	271
Silver — pure	1000	208
Zinc do.	880	183
Gold do.	800	167
Cadmium do.	520	108
Bismuth do.	250	52
Tin do.	130	27
Lead do.	75	16

* This wrought iron was made from the above-mentioned cast iron.

This table exhibits a curious fact, viz., the high degree of hardness of cast iron as compared with that of all other metals, and although we found alloys which possessed an extraordinary degree of hardness, still none were equal to cast iron.

The first series of alloys we shall give is that of copper and zinc.

Formulæ of Alloys and per centages.	Weight employed.	Obtained Cast Iron = 1000.	Calculated* Cast Iron = 1000.
	lbs.		
$Zn\,Cu_5$ { Cu 82·95 / Zn 17·05 }	2050	427·08	280·83
$Zn\,Cu_4$ { Cu 79·56 / Zn 20·44 }	2250	468·75	276·82
$Zn\,Cu_3$ { Cu 74·48 / Zn 25·52 }	2250	468·75	276·04
$Zn\,Cu_2$ { Cu 66·06 / Zn 33·94 }	2270	472·92	261·04
$Zn\,Cu$ { Cu 49·32 / Zn 50·68 }	2900	604·17	243·33
$Cu\,Zn_2$ { Cu 32·74 / Zn 67·26 }	Broke with 1500 lbs. without the point entering.		
$Cu\,Zn_3$ { Cu 24·64 / Zn 75·36 }	Broke with 1500 lbs. with an impression ¼ *mm.* deep.		
$Cu\,Zn_4$ { Cu 19·57 / Zn 80·43 }	Entered a little more than the above; broke with 2000 lbs.		
$Cu\,Zn_5$ { Cu 16·30 / Zn 83·70 }	Entered 2 *mm.* with 1500 lbs.; broke with 1700 lbs.		

These results show that all the alloys containing an excess of copper are much harder than the metals composing them, and what is not less interesting, that the increased degree of hardness is due to the zinc, the softer metal of the two which compose these alloys. The quantity of this metal must, however, not exceed 50 per cent. of the alloy, or the alloy becomes so brittle that it breaks as the steel point penetrates. We believe that some of these alloys, with an excess of zinc, and which are not found in commerce owing to their white appearance, deserve the

* To calculate the hardness of an alloy we multiplied the per centage quantity of each metal by the respective hardness of that metal, added the two results together, and divided by 100. The quotient is the theoretical hardness.

attention of engineers. There is in this series an alloy to which we wish to draw special attention, viz., the alloy Cu Zn composed in 100 parts of

Copper	49·32
Zinc	50·68
	100·00

Although this alloy contains about 20 per cent. more zinc than any of the brasses of commerce, still it is, when carefully prepared, far richer in colour than the ordinary alloys of commerce. The only reason that we can give why it has not been introduced into the market is, that when the amount of zinc employed exceeds 33 per cent. the brass produced becomes so white that the manufacturers have deemed it advisable not to exceed that proportion. If, however, they had increased the quantity to exactly 50·68 per cent. and mixed the metals well, they would have obtained an alloy as rich in colour as if it had contained 90 per cent. of copper, and of a hardness three times as great as that given by calculation. In order to enable engineers to form an opinion as to the value of this cheap alloy we give them the degrees of hardness of several commercial brasses: —

Commercial Brasses.		Weight employed.	Cast Iron = 1000. Obtained.	Cast Iron = 1000. Calculated.
		lbs.		
"Large Bearing"	Copper 82·05 *Tin 12·82 Zinc 5·13	2700	562	259
"Mud Plugs".....	Copper 80 *Tin 10 Zinc 10	3600	750	262
"Yellow Brass"	Copper 64 Zinc 36	2500	520	258
"Pumps and Pipes"	Copper 80·0 *Tin 5·0 Zinc 7·5 Lead 7·5	1650	343	257

* These alloys all contain tin.

The alloy Cu Zn possesses another remarkable property, viz., the facility with which it is capable of crystallising in prisms half an inch in length, of extreme flexibility. There is no doubt that this alloy is a definite chemical compound, and not a mixture of metals, as alloys are generally considered to be. Our researches on the conductibility of heat by alloys, which we have recently presented to the Royal Society, leave no doubt that many alloys are definite chemical compounds.

On Bronze Alloys.

Formulæ of Alloys and per centages.	Weight employed.	Obtained Cast Iron = 1000.	Calculated Cast Iron = 1000.
	lbs.		
$Cu\ Sn_5$ { Cu 9·73 / Sn 90·27 }	400	83·33	51·67
$Cu\ Sn_4$ { Cu 11·86 / Sn 88.14 }	460	95·81	59·56
$Cu\ Sn_3$ { Cu 15·21 / Sn 84·79 }	500	104·17	68·75
$Cu\ Sn_2$ { Cu 21·21 / Sn 78·79 }	650	135·42	84·79
Cu Sn { Cu 34·98 / Sn 65·02 }	At 700 lbs. the point entered one half and the alloy broke.		
$Sn\ Cu_2$ { Cu 48·17 / Sn 51·83 }	At 800 lbs. the alloy broke without the point entering.		
$Sn\ Cu_3$ { Cu 61·79 / Sn 38·21 }	At 800 lbs. the alloy broke into small pieces (blue alloy).		
$Sn\ Cu_4$ { Cu 68·27 / Sn 31·73 }	1300 lbs. divided the alloy into 2 pieces without the point having entered 1 *mm.*		
$Sn\ Cu_5$ { Cu 72·90 / Sn 27·10 }	The same as the preceding.		
$Sn\ Cu_{10}$ { Cu 84·32 / Sn 15.68 }	4400	916·66	257·08
$Sn\ Cu_{15}$ { Cu 88·97 / Sn 11·03 }	3710	772·92	270·83
$Sn\ Cu_{20}$ { Cu 91·49 / Sn 8·51 }	3070	639·58	277·70
$Sn\ Cu_{25}$ { Cu 93·17 / Sn 6·83 }	2890	602·08	279·16

The results obtained from this series of alloys lead to several conclusions deserving our notice. First, the marked softness of all the alloys containing an excess of tin; secondly, the extraordinary fact that an increased

quantity of so malleable a metal as copper should so suddenly render the alloy brittle, for the

Alloy Cu Sn_2

or

Copper ..... 21·21
Tin ..... 78·79 } is not brittle,

whilst the alloy Cu Sn

or

Copper ..... 34·98
Tin ..... 65·02 } is brittle.

Therefore the addition of 14 per cent of copper renders a bronze alloy brittle. This curious fact is observed in all the alloys with excess of copper, Sn Cu_2, Sn Cu_3, Sn Cu_4, Sn Cu_5, until we arrive at one containing a great excess of copper, viz., the alloy Sn Cu_{10}, consisting of copper 84·68 and tin 15·32, when the brittleness ceases; but strange to say this alloy, which contains four-fifths of its weight of copper, is notwithstanding nearly as hard as iron. This remarkable influence of copper in the bronze alloys is also visible in those composed of

Sn Cu_{15}, containing 88·97 of copper.
Sn Cu_{20}, „ 91·49 „
Sn Cu_{25}, „ 93·17 „

Copper acquires such an increased degree of hardness by being alloyed with tin or zinc that we thought it interesting to ascertain if alloys composed of these two metals would also have a greater degree of hardness than that indicated by theory; we accordingly had a series of alloys prepared in equivalent quantities, and these are the results arrived at:

Formulæ of Alloys and per centages of each.	Weight employed.	Obtained Cast Iron = 1000.	Calculated Cast Iron = 1000.
	lbs.		
$Zn\ Sn_2$ { Zn 21·65 / Sn 78·35 }	300	64·50	60·83
$Zn\ Sn$ { Zn 35·60 / Sn 64·40 }	330	68·75	82·70
$Sn\ Zn_2$ { Sn 47·49 / Zn 52·51 }	400	83·33	110·00
$Sn\ Zn_3$ { Sn 37·57 / Zn 62·43 }	450	93·70	124·58
$Sn\ Zn_4$ { Sn 31·14 / Zn 68·86 }	505	105·20	131·22
$Sn\ Zn_5$ { Sn 26·57 / Zn 73·43 }	600	125·00	142·08
$Sn\ Zn_{10}$ { Sn 15·32 / Zn 84·68 }	580	120·83	158·33

These results show that these metals exert no action on each other, as the numbers indicating the degrees of hardness of their alloys are rather less than those required by theory. Our researches on the conductibility of heat by the three above series of alloys throw, we believe, some light on the great difference which the alloys of bronze present as compared with those of tin and zinc; for we have stated above that the latter conduct heat as a mixture of metals would do, and not as the former series, which conduct heat as definite chemical compounds.

We shall conclude by giving the degrees of hardness of two other series of alloys, viz., those composed of lead and antimony, and lead and tin. In the series of lead and tin we find that tin also increases the hardness of lead, but not in the same degree as it does that of copper.

Lead and Antimony.

Formulæ of Alloys and per centages.	Weight employed.	
	lbs.	
$Pb\ Sb_5$ { Pb 24·31 / Sb 75·69 }		Entered 2·5*mm.* with 800 lbs.; then broke.
$Pb\ Sb_4$ { Pb 28·64 / Sb 71·36 }		Entered 2·7*mm.* with 800 lbs.; broke with 900 lbs.
$Pb\ Sb_3$ { Pb 34·86 / Sb 65·14 }	875	
$Pb\ Sb_2$ { Pb 44·53 / Sb 55·47 }		Entered 2·5*mm.* with 500 lbs.; broke with 600 lbs.
$Pb\ Sb$ { Pb 61·61 / Sb 38·39 }	500	
$Sb\ Pb_2$ { Pb 76·32 / Sb 23·68 }	385	
$Sb\ Pb_3$ { Pb 82·80 / Sb 17·20 }	310	
$Sb\ Pb_4$ { Pb 86·52 / Sb 13·48 }	300	
$Sb\ Pb_5$ { Pb 88·92 / Sb 11·08 }	295	

Lead and Tin.

Formulæ of Alloys and per centages.	Weight employed.	Obtained Cast Iron = 1000.	Calculated Cast Iron = 1000.
	lbs.		
$Pb\ Sn_5$ { Pb 26·03 / Sn 73·97 }	200	41·67	23·96
$Pb\ Sn_4$ { Pb 30·57 / Sn 69·43 }	105	40·62	23·58
$Pb\ Sn_3$ { Pb 36·99 / Sn 63·01 }	160	32·33	22·83
$Pb\ Sn_2$ { Pb 46·82 / Sn 53·18 }	125	26·04	20·09
$Pb\ Sn$ { Pb 63·78 / Sn 36·22 }	100	20·83	19·77
$Sn\ Pb_2$ { Pb 77·89 / Sn 22·11 }	125	26·04	18·12
$Sn\ Pb_3$ { Pb 84·09 / Sn 15·91 }	135	28·12	17·23
$Sn\ Pb_4$ { Pb 87·57 / Sn 12·43 }	125	26·04	17·08
$Sn\ Pb_5$ { Pb 89·80 / Sn 10·20 }	110	22·92	16·77

We have great pleasure in thanking here Mr. Siméon Stoikowitsch, F.C.S., for his valuable assistance during these long researches.

VIII. — *On a Yellow Colouring Matter obtained from the Leaves of the Polygonum Fagopyrum, or Common Buckwheat.*

By Edward Schunck, Ph. D., F.R.S.

Read October 6th, 1857.

Among the many plants which have been supposed to contain or yield indigo-blue, the *Polygonum Fagopyrum* or common buckwheat, a plant extensively cultivated in some countries for the sake of its seed, which is used as an article of food, is mentioned by some authors. In the *Mechanic's Magazine* for November 1830, for instance, the following directions are given for obtaining a blue colour from this plant: — "Take the buckwheat out of the ground before the seed has become quite hard and lay it on the ground in the sun, until it is dry. Then throw the plant into heaps, moisten it and allow it to ferment, until decomposition commences, when it will assume a blue colour. It must now be formed into cakes, which may be dried either in the sun or in a stove. It imparts to boiling water a blue colour, which is not changed either by acetic or sulphuric acid. A durable blue may be dyed with it." Though it does not follow from this that the blue colouring matter thus obtained was really indigo-blue, still it seemed not improbable that it might be identical with the latter, since another species belonging

to the same genus, the *Polygonum tinctorium*, is remarkable for the large amount of indigo-blue which it affords. Nevertheless on submitting the leaves of the plant to the same process as that previously employed in the case of the *Isatis tinctoria* and other plants producing indigo, I was unable to obtain a trace of that or any other blue colouring matter. The examination led, however, to the discovery of a crystallised yellow colouring matter, the method of preparation and properties of which I shall now proceed to describe.

The plant, which is one easily cultivated and will grow in the poorest soils, having been allowed to attain its full size, the leaves are separated from the stalks, and treated for some time with boiling water. The decoction, which is muddy and of a greenish colour, is strained through calico, and the mass remaining behind is well pressed in order to remove all the liquid. A little acetate of lead is now added to it. This produces a bulky yellowish-green precipitate, consisting of chlorophyll and other impurities in combination with oxide of lead. Care must be taken to add only so much acetate of lead as to render the supernatant liquid clear and transparent, as an excess would precipitate a portion of the colouring matter. The liquid, having been again raised to the boiling point, is filtered boiling. It has a fine golden yellow colour, but on being mixed with a quantity of acetic acid it becomes pale yellow, and on being now allowed to stand for some time it deposits a voluminous mass of small yellow crystalline needles. These are collected on a filter, slightly washed with cold water and then re-dissolved in boiling water, to which a little acetic acid is added. A small quantity of white matter is usually left undissolved, which is separated by filtration. The filtered liquid again deposits, on cooling and standing, a mass of crystals, which, after being col-

lected on a filter and washed with cold water, are dissolved in boiling alcohol. The alcoholic solution is filtered from a small quantity of insoluble matter, and then distilled until the greatest part of the spirit has passed over. The dark yellow liquid left in the retort is then poured into a dish, and allowed to stand for some time, when it deposits the pure colouring matter as a mass of crystalline needles of a pale primrose-yellow colour. These are collected on a filter, washed with a little cold alcohol, and allowed to dry spontaneously. Its properties are as follows: —

It is perfectly tasteless, and its solutions are neutral to test paper. When heated on platinum foil it melts to a brown transparent liquid, and then burns with a yellow flame, leaving much charcoal, which, on being heated, burns slowly away without leaving any ash. When heated in a tube it melts, gives off fumes having a strong empyreumatic smell, and yields a yellow oily sublimate, in which nothing crystalline is formed even after several days. It is hardly soluble in cold water, and only sparingly soluble in boiling water. The boiling watery solution deposits it on cooling in yellow silky needles, which, when dry, form a compact silky mass. It is more easily soluble in boiling alcohol than in water. It does not separate from the alcoholic solution on cooling, but is left on evaporation in star-shaped masses of a darker yellow colour than the needles obtained from the watery solution. Strong muriatic acid changes its colour to a deep yellow without decomposing it. When concentrated sulphuric acid is brought into contact with it, the acid acquires at first a greenish colour, and then dissolves it entirely, forming a deep yellow solution, from which the greatest part of the colouring matter is precipitated by water in yellow flocks. If, however, the solution in acid be heated it becomes black and gives off sulphurous acid in abundance, the colouring matter being

decomposed. When the substance is dissolved in boiling water to which a little sulphuric acid has been added, and the solution is boiled for some time, no decomposition apparently takes place, proving that this colouring matter is not, like some others, a copulated compound. Nitric acid of ordinary strength dissolves it even in the cold, forming a dark orange-coloured liquid, which, on being boiled, evolves nitrous acid, and becomes pale yellow; the liquid on evaporation yielding a large quantity of oxalic acid. When suspended in water, and exposed to the action of chlorine gas, it gradually dissolves, forming a brownish-yellow solution, which on evaporation leaves a brownish-yellow glutinous residue. This residue has an astringent taste, which is, however, partly disguised by the muriatic acid formed during the decomposition; its watery solution gives with gelatine a curdy precipitate, similar to that produced with the latter by tannin. The colouring matter dissolves easily in caustic potash and soda, liquid ammonia, baryta water and lime water, forming deep yellow solutions, which, on the addition of an excess of acid, become pale yellow, and on standing again deposit the colouring matter in pale yellow needles. If, however, these solutions be left to stand for some time exposed to the air, a decomposition of the colouring matter seems to take place, for on now adding an excess of acid, the solutions remain yellow, and deposit no crystals. If the ammoniacal solution be left exposed to the atmosphere for some time and then evaporated, it leaves a residue consisting of crystals of colouring matter mixed with an amorphous substance. On adding water to the residue the latter dissolves, and the solution, after being filtered from the crystals and evaporated, leaves a transparent, yellow, brittle mass resembling gum, which contains only a trace of ammonia, and is probably a

product of decomposition formed by the action of the atmospheric oxygen on the colouring matter.

The watery solution produces, with different reagents, the following reactions. With acetate of alumina it gives a bright yellow flocculent precipitate; the filtered liquid is still very yellow, but on the addition of ammonia it deposits some more yellow precipitate, and then appears colourless. With protosulphate of iron it turns of a greenish colour, but on standing exposed to the air the colour changes to a dark green, whilst a dark green powder is deposited; the addition of ammonia now produces a dark brown precipitate, the liquid becoming colourless. On the addition of perchloride of iron it turns of a brownish-olive colour. Both the watery and the alcoholic solution give, with acetate of lead, a precipitate of a bright yellow colour, inclining to orange, which very much resembles that of chromate of lead; the filtered liquid in either case still retains some of its yellow colour. With basic acetate of lead the watery solution gives a precipitate of a rather darker yellow colour than that with neutral acetate, the filtered liquid being colourless. The watery solution when cold becomes, on the addition of acetate of copper, greenish-yellow, and remains clear, but on being boiled it deposits an abundant greenish-yellow precipitate, which dissolves again almost entirely when the liquid cools. The watery solution gives no precipitate with nitrate of silver, but on standing it becomes muddy and black, and deposits a fine black powder. On the addition of chloride of gold it also soon becomes muddy, and deposits bright spangles of metallic gold; when caustic soda is added at the same time, the reduction takes place instantaneously, the gold being deposited partly as a black powder, partly as a metallic mirror covering the sides of the glass. Protochloride of tin produces in the watery solution a bright

yellow precipitate, the supernatant liquid remaining muddy for a long time.

Printed calico, when immersed in the warm watery solution of this colouring matter, becomes fully dyed, the alumina mordant acquiring a dark yellow colour, the tin mordant a light yellow, and the iron mordant various shades of yellowish-brown according to the strength of the mordant employed. Silk and wool do not acquire any colour in the boiling watery solution, unless they have been previously prepared with some mordant.

The substance contains no nitrogen, for on being treated with boiling caustic soda lye, or heated in a tube with soda-lime, it evolves no ammonia. Its composition was determined by the following analyses: —

I. 0·4145 grm. of the substance, crystallised from alcohol, dried at 100°C. and burnt with chromate of lead, gave 0.7550 grm. carbonic acid and 0·2200 water.

II. 0·4085 grm. crystallised from water, gave 0·7450 grm. carbonic acid and 0·2130 water.

III. 0·4205 grm. of another preparation gave 0·7715 grm. carbonic acid and 0·2255 water.

IV. 0·4170 grm. gave 0.7640 grm. carbonic acid and 0·2225 water.

These numbers correspond in 100 parts to

	I.	II.	III.	IV.
Carbon.....	49·67	49·73	50·03	49·96
Hydrogen.....	5·89	5·79	5·95	5·92
Oxygen.....	44·44	44·48	44·02	44·12
	100·00	100·00	100·00	100·00

The lead compound was prepared by adding to the alcoholic solution an alcoholic solution of acetate of lead. The dark yellow precipitate was collected on a filter,

washed with cold alcohol, and dried over sulphuric acid. It yielded on being analysed the following results:—

0·9555 grm. gave 1·1315 grm. carbonic acid and 0·3090 water.

0·7630 grm. gave 0·4025 grm. sulphate of lead.

From these numbers may be deduced the following composition:—

	Equivs.		Calculated.	Found.
Carbon	30	180	31·83	32·29
Hydrogen	18	18	3·18	3·59
Oxygen	18	144	25·48	25·31
Oxide of Lead	2	223·4	39·51	38·81
		565·4	100·00	100·00

If the formula of the lead compound is $C_{30}\ H_{18}\ O_{18} + 2\ Pb\ O$, as the above analysis seems to indicate, then that of the substance in its uncombined state must be $C_{30}\ H_{20}\ O_{20}$ and its theoretical composition will be as follows:—

	Equivs.		
Carbon	30	180	50·00
Hydrogen	20	20	5·55
Oxygen	20	160	44·45
		360	100·00

A comparison of the properties and composition of this substance with those of *Rutine* or *Rutic Acid*, the colouring matter discovered by Weiss* in the *Ruta graveolens* or common rue, and by Rochleder and Hlasiwetz† in capers, leads to the conclusion that they are identical. A colouring matter, having the same properties and composition, and being apparently the same substance, has also lately been

* Pharmaceut. Centralblatt, 1842, s. 903.

† Annalen d. Chemie u. Pharm., Bd. LIII. s. 385.

discovered by Moldenhauer,* in the leaves of the *Ilex aquifolium*, or common holly, and called by him *Ilixanthine*. The quantity to be obtained from the buckwheat leaves seems, however, to be much more considerable than that procurable from any other source. In one experiment I found that 30lbs. of fresh leaves yielded 240 grains of crystallised rutine, or a little more than the thousandth part of their weight. In countries where the plant is cultivated it might be worth while to collect the leaves as a dyeing material, since the seeds are the only part at present employed for any useful purpose.

* Annalen d. Chemie u. Pharm., Bd. CII. s. 346.

IX. — *Researches in the Higher Algebra.*

By James Cockle, M.A., F.R.A.S., F.C.P.S.,
Barrister-at-Law.

Communicated by the Rev. R. Harley, F.R.A.S.

Read October 5th, 1858.

§ 1.

The higher algebra conducts us to results, the complexity of which is unfavourable, not only to the progress of research, but sometimes perhaps to intelligibility. The object of this paper is to exhibit some of them in a form which, without detracting from their real generality, will render their import comparatively easy of recognition.

§ 2.

Let i be an unreal fifth root of unity, and let

$i+i^4=\phi(i)$, $i^2+i^3=\phi(i^2)$, $3+\phi(i^2)=a$, $1+2\phi(i^2)=b$,
$1+2\phi(i)=c$, $3+\phi(i)=d$,

then

$$\phi(i)+\phi(i^2)=\phi(i)\cdot\phi(i^2)=-1,\ b+c=0, \text{ and}$$
$$ad=b^2=c^2=-bc=5.$$

§ 3.

Again, let v, w, x, y and z be the roots of a given equation in u of the fifth degree, and let

$v+iw+i^2x+i^3y+i^4z=f(i)$, $f(i)\cdot f(i^2)\cdot f(i^3)\cdot f(i^4)=\theta$ and
$v^2(wz+xy)+w^2(vx+yz)+x^2(vz+wy)+y^2(vw+xz)$
$+z^2(vy+wx)=\mathrm{U}=\psi(u)$,

then, if the given equation be of the form

$$u^5 - 5Qu^3 + E = 0 \quad \ldots\ldots\ldots\ldots \quad (1),$$

there subsists the relation

$$\theta + 5U = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (2).$$

§ 4.

When one of the six values of θ vanishes, (1) admits of finite algebraic solution.

It is unnecessary to repeat the investigations in the course of which I arrived at this criterion of solvibility, or to exhibit the complicated formula by which, in the general case, it is expressed.* But the solution, if any exist, of the general equation of the fifth degree, will contain a term of the form $i^n\Theta$, where Θ vanishes with θ.

§ 5.

When (1) holds, the conditions of homogeneity indicate

$$\theta^6 + aQE\theta^4 + \beta Q^4\theta^3 + \gamma Q^2E^2\theta^2 + (\delta Q^5E + \epsilon E^4)\theta + \zeta Q^3 + \eta Q^3E^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

as the form of the equation in θ. And in determining a, β, &c., we may make $\eta = 0$, for E^3 does not enter into the criterion alluded to.

§ 6.

Let $\psi(u - h)$ or V be the value of U when each of the roots of (1) is diminished by h. Then

$$V = U - DU \cdot h + D^2U \cdot \frac{h^3}{1 \cdot 2} - D^3U \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

where D is the differential symbol $\frac{d}{dv} + \frac{d}{dw} + \frac{d}{dx} + \frac{d}{dy} + \frac{d}{dz}$.

But

$$DU = \Sigma v^2 w + 2\Sigma vwx = -5Q, \quad D^2U = 4\Sigma v^3 + 10\Sigma vw = 0,$$

$$D^3U = 48\Sigma v = 0, \quad D^4U = 240,$$

consequently

$$U = V - 5Q \cdot h - 10 \cdot h^4 \quad \ldots\ldots\ldots\ldots \quad (4).$$

* See *Phil. Mag.*, May 1858, p. 389.

§ 7.

Next, let (1) become

$$u^5 - \frac{5}{2}u^3 + \frac{3}{2} = 0 \quad \ldots\ldots\ldots\ldots \quad (5),$$

which is equivalent to

$$(u-1)^2\left(u^3 + 2u^2 + 3u + \frac{3}{2}\right) = 0,$$

or, if $u - 1 = p$, to

$$p^2\left(p^3 + 5p^2 + 10p + \frac{15}{2}\right) = 0 \quad \ldots\ldots\ldots \quad 6).$$

Then, if q, r and s be the roots of

$$p^3 + 5p^2 + 10p + \frac{15}{2} = 0 \quad \ldots\ldots\ldots\ldots \quad (7),$$

we have

$$\mathrm{U} - q^2rs + \frac{25}{2} = 0 \quad \ldots\ldots\ldots\ldots \quad (8).$$

§ 8.

For, since h is arbitrary, we may assume it equal to unity, and replace (4) by

$$\mathrm{U} = \psi(u-1) - 5\mathrm{Q} - 10,$$

a relation which, when (5) is satisfied, becomes

$$\mathrm{U} = \psi(p) - \frac{25}{2}.$$

But, by (6), two of the values of p are zero, and, therefore,

$$\psi(p) = q^2rs,$$

and (8) is verified.

§ 9.

Permuting q, r and s in (8), and determining the symmetrical functions by means of (7), we find that the six values of U are the roots of

$$\left\{\mathrm{U}^3 + \frac{3\cdot 5^3}{2^3}\mathrm{U} - \frac{5^4}{2^4}\right\}^2 = 0.$$

Consequently, by (2), the corresponding equation in θ is

$$\left\{\theta^3 + \frac{3\cdot 5^5}{2^3}\theta + \frac{5^7}{2^4}\right\}^2 = 0 \quad \ldots\ldots\ldots\ldots \quad (9).$$

§ 10.

When $$Q=\frac{1}{2} \text{ and } E=\frac{3}{2}$$

(5) is identical with (1), and (9) with (3). Therefore, developing (9) and comparing coefficients, we have

$$\frac{3a}{2^3}=\frac{3\cdot 5^5}{2}, \text{ or } a=2\cdot 5^5; \quad \frac{\beta}{2^4}=\frac{5^7}{2^3}, \text{ or } \beta=2\cdot 5^7;$$

$$\frac{3^2\gamma}{2^4}=\frac{3^3\cdot 5^{10}}{2^4}, \text{ or } \gamma=5^{10}; \quad \frac{3\delta}{2^6}+\frac{3^4\epsilon}{2^4}=\frac{3\cdot 5^{12}}{2^5}; \quad \frac{\zeta}{2^8}=\frac{5^{14}}{2^5}, \text{ or } \zeta=5^{14}.$$

§ 11.

In (1) let $Q=0$ and $E=-1$ and replace v, w, x, y, z, by $1, i, i^2, i^3, i^4$, respectively. Then, for all integral values of m, $f(i^m)=0$ and θ vanishes. But if we make $v=i$, $w=1$, the other roots remaining undisturbed, we find

$$f(i)=ai, \; f(i^2)=bi^4, \; f(i^3)=ci^2, \; f(i^4)=d,$$

and $\theta=abcdi^8=-5^2i^8$. Consequently the equation in θ will, on the present supposition, be

$$\theta(\theta^5+5^{10})=0 \quad \ldots\ldots\ldots \quad (10),$$

whence, comparing (10) and (3),

$$\epsilon=5^{10}$$

and, by the last section,

$$\delta=2\cdot 5^{12}-2^3 3^3\epsilon=2\cdot 5^{10}(5^2-2\cdot 3^3)=-58\cdot 5^{10}.$$

§ 12.

The equation in θ is, therefore,

$$\theta^6+2QE5^5\theta^4+2Q^4 5^7\theta^3+Q^3E^2 5^{10}\theta^2$$
$$-(58Q^5-E^3)E5^{10}\theta+5^{14}Q^3=0 \quad \ldots\ldots\ldots \quad (11)$$

§ 13.

We may obtain (9) from (5) thus; let

$$f(i)=1+i+i^2x+i^3y+i^4z,$$

then x, y and z are the roots of

$$x^3+2x^2+3x+\frac{3}{2}=0 \quad \ldots\ldots\ldots \quad (12),$$

and, by means of symmetric functions, we are conducted to

$$f(i)\cdot f(i^4)=2c+b(xz+y),\ f(i^2)\cdot f(i^3)=2b+c(xz+y).$$

It follows from this, that

$$\theta=-5(xz+y-2)^2$$

or, eliminating xz and reducing by the aid of (12),

$$\theta-5^2\left(\frac{3}{2}y+1\right)=0,$$

whence, permuting and determining the symmetric functions by means of (12),

$$\theta^3+\frac{3\cdot5^5}{2^3}\theta+\frac{5^7}{2^4}=0,$$

which verifies (9).

§ 14.

Again, let $5Q=1$ and $E=0$ and replace v, w, x, y, z by 1, j, j^2, 0, 0 respectively, j being an unreal cube root of unity. Then

$$f(i)\cdot f(i^4)=bj^2,\ f(i^2)\cdot f(i^3)=cj^2,$$

and, consequently,

$$\theta=bcj,\ \text{or}\ \theta+5j=0,$$

whence

$$(\theta^3+5^3)^2=0.$$

Developing this equation, and comparing its coefficients with those of (3), we have

$$2\cdot5^3=\frac{\beta}{5^4},\ \text{or}\ \beta=2\cdot5^7$$

$$5^6=\frac{\zeta}{5^8},\ \text{or}\ \zeta=5^{14},$$

as before.

§ 15.

The equation (11) may be put under the form

$$(\theta^3+5^5QE\theta+5^7Q^4)^2=5^{10}(108Q^5E-E^4)\theta,$$

and this may be readily transformed into

$$(\Im^6+A\Im^2+B)^2=C^2\Im^2,$$

or

$$\vartheta^5+A\vartheta^3\pm C\vartheta+B=0 \quad \ldots\ldots\ldots\ldots (13),$$

where

$$C^2=\frac{108AB}{5^3}-\frac{A^4}{5^3B} \quad \ldots\ldots\ldots\ldots (14).$$

§ 16.

Determining λ by the equation $A=B\lambda^3$, and assuming $A\lambda^4=B\lambda^6=5H$, we obtain from (14)

$$C^2=\frac{108(A\lambda^4)(B\lambda^6)}{5^3\lambda^{10}}-\frac{(A\lambda^4)^4}{5^3(B\lambda^6)\lambda^{10}},$$

or

$$C^2\lambda^{10}=(C\lambda^5)^2=108H^3-H^3.$$

Consequently, when (14) is satisfied, (13) may, by means of the assumption $t=\lambda\vartheta$, be transformed into

$$t^5+5Ht^3\pm\sqrt{108-H}\cdot Ht+5H=0 \quad \ldots\ldots (15),$$

or

$$\{t^5+5H(t^3+1)\}^2=(108-H)H^2t^3 \quad \ldots\ldots (16),$$

or, making $t^3=s$,

$$\{s^3+5H(s+1)\}^2=(108-H)H^2s \quad \ldots\ldots (17).$$

§ 17.

If, therefore, we determine T and F from the equations

$$5^4TF=H=5^6T^4,$$

which in fact lead to

$$F=5^2T^3,$$

the solution of (17) is made to depend upon that of

$$\xi^5-5T\xi^3+5^2T^3=0 \quad \ldots\ldots\ldots (18),$$

a form into which the assumptions

$$5^2Q^3\mu^4=E, \ \mu\xi=u$$

suffice to convert (1).

§ 18.

The following remarks arise out of this discussion: —

1°. The functions a, b, c and d seem to be entitled to rank as canonical expressions in the theory of equations of the fifth degree.

2°. Since Sir W. Rowan Hamilton's "Inquiry into the Validity of Mr. Jerrard's Method" has established the fact that the general equation may be reduced to the form (1), that form imposes only a seeming, not a real, restriction on the generality of our investigations.

3°. The same remark applies, when C is unrestricted in value, to (13); for to such a form the general equation of the sixth degree may be reduced by Mr. Jerrard's method.

4°. The condition (14) appears to restrict the generality of the sextic. But the solution of the restricted sextic is reduced to that of (1), or, which amounts to the same thing, the solution of (15), (16) or (17) is reduced to that of (18.)

5°. Were it possible to solve (15), (16) or (17) by means of equations of degrees lower than the fifth, we should probably be advancing a step towards clearing up the difficulties which encircle the question of the finite algebraic solution of (1), and, therefore, of the general quintic.

6°. If the roots of (15), (16) or (17) are connected, symmetrically or otherwise, with those of the final sextic in a proposed method of solution of a quintic, the solvibility or insolvibility of those equations may afford a test, perhaps the simplest which the case admits, of the sufficiency of the proposed method.

§ 19.

Mr. Harley, in whose hands I was so fortunate as to place these investigations, has since verified all the coefficients of the equation in θ, with the exception of the last two. He has effected this by a direct, original and general process, of independent interest and of intrinsic value and importance. His results, perhaps carried still further, will, I hope, be soon laid before the Society.

§ 20.

It is easy to evolve all the roots of

$$u^5 - 5Mu^4 - 5Pu^3 - 5Qu^2 - 5Ru + E = 0 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

from a single expression, if unsymmetric functions are employed — thus

$$\frac{1}{5}\left\{i^m f(i) + i^{2m} f(i^2) + i^{3m} f(i^3) + i^{4m} f(i^4) + i^{5m} f(i^5)\right\}$$

may, by assigning appropriate values to m, be made to yield all the roots in succession; but the possibility of evolving them from an expression of the form

$$M + i^m \Theta^{\mathrm{i}} + i^{2m} \Theta^{\mathrm{ii}} + i^{3m} \Theta^{\mathrm{iii}} + i^{4m} \Theta^{\mathrm{iv}}$$

where Θ^{i}, Θ^{ii}, Θ^{iii}, Θ^{iv}, are functions of the coefficients of (19) is by no means an obvious consequence.

§ 21.

But, when any one of the six values of θ vanishes, all the roots of (19) are included in an expression of the form

$$M + i^m \Theta' + i^{2m} \Theta'' + i^{(3+r)m} \Theta'''$$

where m, as in the last section, is an integer, r is 0 or 1 according to circumstances, and Θ', Θ'', Θ''', are all known or ascertainable functions of the coefficients of (19).

§ 22.

If, then, by any practicable transformation of (19), θ could be made to vanish, all the roots of the transformed, and consequently of the original, equation would be determined.

§ 23.

There are, however, other modes of proceeding. Superadding relations obtained from other sources — the second form of Euler, for example — we may examine the conditions which the evanescence of θ discloses, and endeavour to express u as a function of θ. Thus, in the elementary case in which (19) takes a binomial form, assuming

$$u=\chi(\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6),$$

we find

$$u=\sqrt[5]{-\mathrm{E}}=\sqrt[5]{-\sqrt[4]{\mathrm{E}^4}}=\sqrt[5]{-\sqrt[4]{\frac{-\Sigma\theta_1\theta_2\theta_3\theta_4\theta_5}{5^{10}}}}$$

and, hence, conclude that

$$\Sigma\theta_1\theta_2\theta_3\theta_4\theta_5,\ \text{or}\ \theta_1\theta_2\theta_3\theta_4\theta_5,$$

is a constituent of χ. If we were conducted to unsymmetric or, by the assumption $u=\mathrm{F}(\theta)$, to general functions of θ, it would become necessary to inquire into the solvibility of the equation in θ.

§ 24.

A third mode, also involving that inquiry, would be to seek the values of u in terms of those of θ by elimination. And here, as it would not be difficult to show, Mr. Harley's investigations, as yet unpublished, substantially change the data into four quadratics, involving four unknowns, but with certain relations existing among the coefficients.

§ 25.

Writing for a moment $\theta(u)$ in place of θ, the transition from $\theta(u)$ to $\theta\{\mathrm{R}(u)\}$, where $\mathrm{R}(u)$ is a rational and general or unique function of u — *i.e.* a rational function of any and only one value of u — though perhaps interesting, does not seem to demand our attention at present.

§ 26.

A finite algebraic solution of the general equation of the fifth degree may be impossible or unattainable. If the hypothesis of its existence lead to contradictory or inconsistent conditions, or to a self-contradictory result, such a solution is impossible or unconditionally unattainable. If the hypothesis conduct to the conclusion that every known process for solution leads to an illusory result, *e.g.* to vanishing fractions unassignable in value, or

to equations which present as much or more difficulty than the given one, then the solution is, not absolutely, but conditionally or relatively, unattainable. The latter conclusion may afford probable evidence that the solution is impossible; but the impossibility cannot be said to be demonstrated.

§ 27.

Let

$$v^m + iw^m + i^2x^m + i^3y^m + i^4z^m = f^m(i),$$
$$f^m(i)f^m(i^4) = \tau_1(u^m), \text{ and}$$
$$f^m(i^2)f^m(i^3) = \tau_2(u^m).$$

Then

$$\tau_1(u^m) + \tau_2(u^m) = 2\Sigma v^{2m} - \Sigma v^m w^m,$$
$$-5\tau_1(u^m) \times \tau_2(u^m) = \theta(u^m) + \mathrm{K}$$

where $\theta(u^m)$ is the same function of u^m that θ or $\theta(u)$ is of u, and K is known, and, consequently, τ is known when θ is known. The function $\theta(u^m)$ is known whenever $\theta(u)$ or θ is known.

§ 28.

Let

$$f^m(i)f^n(i^4) + f^m(i^4)f^n(i) = \mathrm{I},$$
$$f^m(i^2)f^n(i^3) + f^m(i^3)f^n(i^2) = \mathrm{J},$$

then

$$\mathrm{I} + \mathrm{J} = 4\Sigma v^{m+n} - \Sigma v^m w^n,$$

and

$$\mathrm{IJ} = \frac{1}{2}\left\{(\mathrm{I}+\mathrm{J})^2 - (\mathrm{I}^2 + \mathrm{J}^2)\right\}.$$

But

$$\mathrm{I}^2 + \mathrm{J}^2 = \{f^m(i)f^n(i^4)\}^2 + \{f^m(i^4)f^n(i)\}^2 + \{f^m(i^2)f^n(i^3)\}^2$$
$$+ \{f^m(i^3)f^n(i^2)\}^2$$
$$+ 2\{f^m(i)f^m(i^4)f^n(i)f^n(i^4) + f^m(i^2)f^m(i^3)f^n(i^2)f^n(i^3)\}$$
$$= (\Sigma) + 2\{\tau_1(u^m)\,\tau_1(u^n) + \tau_2(u^m)\,\tau_2(u^n)\}$$

where (Σ) is a function of τ. Consequently I and J are known when τ, which depends upon θ, is known.

§ 29.

The relations

$$f^m(i)f^n(i^4)+f^m(i^4)f^n(i)=\mathrm{I},$$
$$f^m(i)f^n(i^4)\times f^m(i^4)f^n(i)=\tau_1(u^m)\,\tau_1(u^n),$$

enable us to determine $f^m(i)f^n(i^4)$ and $f^m(i^4)f^n(i)$ in terms of I and τ_1. Similar relations hold for J and τ_2.

§ 30.

Now make $\dfrac{1}{v+p}+\dfrac{i}{w+p}+\dfrac{i^2}{x+p}+\dfrac{i^3}{y+p}+\dfrac{i^4}{z+p}=0.$

This, on reduction, becomes

$$p^5-\frac{f^2(i)}{f(i)}p^3+\frac{f^3(i)}{f(i)}p+\left\{5\mathrm{Q}-\frac{f^4(i)}{f(i)}\right\}=0,$$

and the relation

$$\frac{f^m(i)}{f^n(i)}=\frac{f^m(i)f^n(i^4)}{\tau_1(u^n)}$$

will enable us to express p in terms of I and τ_1.

§ 31.

When, therefore, θ is known, p can be determined; and, if in (1) we substitute $\dfrac{1}{t}-p$ for u and reduce the result to the usual form, the equation in t will be solvible, for a factor of the symmetric product vanishes. The solution of (19) may be made to depend upon that of (1), and the latter, in its turn, upon that of (11) and of the equation in t.

§ 32.

Mr. Harley has discovered that in certain cases U becomes a square, and equals τ^2. This circumstance may throw a light upon the question of the solvibility of the equation in θ.

NOTE. — Mr. Cockle's original communication, consisting of the

first 18 §§, is dated May 17th, 1858. The 19–26 §§ are added by a postscript under date of September 10th, 1858; and §§ 27-32 are added by another postcript dated September 22nd, 1858.

X. — *Note on Dalton's Determination of the Expansion of Air by Heat.*

By J. P. Joule, LL.D., F.R.S., &c.

Read November 2nd, 1858.

In the twenty-first volume of the *Memoirs of the Academy of Sciences*, p. 23, Mr. Regnault, in the course of a discussion of the various co-efficients given for the expansion of air, by different experimenters, remarks that Rudberg had brought to recollection an observation made by Gilbert, in his *Annals*, to the effect that the experiments of Dalton and Gay Lussac, which had been considered as giving almost identical results, differed on the contrary very considerably from one another. Then, referring to Dalton's experiment, related in the *Manchester Memoirs*, vol. v. part ii. p. 599, Regnault shows that if 1,000 measures of air at 55° Fahrenheit expand to 1,325 at 212°, 1,000 measures taken at 32° will become 1,391 at 212°. Upon this he goes on to remark that Dalton did not appear to have been aware of the error which had crept into his calculations, for he says in his *New System of Chemical Philosophy* that the volume of air, according to Gay Lussac's and his own experiments, being taken 1,000 at 32° becomes 1,376 at 212°.

On reading the remarks of the eminent French physicist, the extreme improbability that a man so notoriously exact and careful in his mathematical and arithmetical computations as Dalton should have made the gross error

imputed to him, at once occurred to me. I therefore, on consulting Dalton's works, was not surprised to find that his commentators had entirely misunderstood the facts of the case. These are as follow: — Dalton, in his *Experimental Essays*, read before this Society in the month of October 1801, describes experiments on the expansion of air by heat, the results of which, referred to the freezing point, are accurately stated by Regnault. But in the *New System of Chemical Philosophy* published in 1808, under the article *Temperature*, Dalton, while explaining his *New Table of Temperature*, writes — "The volume at 32° is taken 1,000, and at 212°, 1,376 according to Gay Lussac's and my own experiments. As for the expansion at intermediate degrees, General Roi makes the temperature at mid-way of total expansion, 116½° old scale; from the results of my former experiments (*Manch. Mem.*, vol. v. part ii. p. 599) the temperature may be estimated at 119½°; but I had not then an opportunity of having air at 32°. By more recent experiments I am convinced that dry air of 32° will expand the same quantity from that to 117° or 118° of common scale, as from the last term to 212°." The first part of the above extract contains the passage quoted by Regnault, but its meaning is obviously not that which he infers. The experiments which Dalton states to agree with Gay Lussac's are clearly some unpublished ones made subsequently to those described in the *Manchester Memoirs*. He nowhere that I can discover advances the assertion, attributed to him by Gilbert and adopted by Rudberg and Regnault, that his former "experiments" agree exactly with those of Gay Lussac. They were, however, highly important at the time when they were made, and justified the approximately correct conclusion he drew, that *all elastic fluids under the same pressure expand equally by heat.*

Dalton was at once aware of the immense importance of

this law, and in a sentence prophetic of the advancement of the theory of heat in recent times, pointed to the force of heat as the *sole* and immediate source of expansion in elastic fluids, and predicted that a study of their phenomena would ultimately lead to general laws respecting the absolute quantity and the nature of heat.

XI.—*On the Utilization of the Sewage of London and other large towns.*

By J. P. Joule, LL.D., F.R.S., &c.

Read November 30th, 1858.

I have learned with regret that a system of metropolitan drainage has been adopted, and is about to be attempted by the Metropolitan Board of Works, which I consider to be a stride in the wrong direction, and which, if persevered in and copied by other towns, must be fraught with disastrous consequences to the national prosperity. I cannot however say that I felt much surprised at the intelligence, as I knew that the advice and assistance of scientific men had not been sought, except in one or two solitary instances, and that the professional engineers consulted had been limited to only a few, however eminent, individuals, who differ among themselves as to the contemplated works. On a question of such vital importance as that which has been raised, I think all ought to contribute what experience, information, or common sense they possess, and so form a concentrated expression of opinion which cannot be disregarded. I therefore, though the subject is somewhat new to me, hesitate not to introduce it to the Society, in the hope of eliciting the opinions of those who may have studied it better than myself.

From the *Report** of Messrs. Hawksley, Bidder and

* *Report* presented to the Metropolitan Board of Works, ordered by the House of Commons to be printed 13th July, 1858.

Bazalgette, I find that the history of the present question of London sewage dates from the year 1847, when, instead of the eight separate commissions which had previously existed, a consolidated one was appointed. This was shortly followed by a second, which advertised for and obtained 116 plans. A third and fourth commission reported against those plans, and appointed Mr. Forster as their engineer, who, after preparing a plan for the drainage of the north district of the Thames, died in consequence of the anxieties of his position. A fifth commission was embarrassed by the plans of the "Great London Drainage Company," which, after occupying a great part of the session of 1853, were ultimately rejected by parliament. In 1854 Messrs. Bazalgette and Haywood prepared a scheme, but another proposal by Mr. Ward having received the sanction of the Secretary of State, a sixth commission was appointed, which invited plans but arrived at no conclusions. In 1856 the Metropolitan Board instructed their engineer to report and prepare plans. Sir B. Hall proposed modifications which were adopted. This final plan was submitted to three referees, viz., Captain Galton R.E., James Simpson C.E., and Thomas E. Blackwell C.E., who reported thereon in July, 1857,* and, on their report being objected to by the Board, suggested material modifications of the plans proposed in the report submitted to parliament by Her Majesty's first Commissioner of Works. A further communication from Messrs. Galton and Simpson, involving a third plan with further modifications, was made in January and February, 1858. Finally, at the request of the Metropolitan Board, Messrs. Bidder, Hawksley and Bazalgette reported on the 6th of April, 1858, on the plans of the government referees as from time to time modified. This last report, which in

* *Report on Metropolitan Drainage*, ordered by the House of Commons to be printed 3rd August, 1857.

general opposes the plans of the referees, appears to be the one finally adopted by the Board of Works on June the 29th, 1858.

In justice to the eminent engineers I have named it is needful to premise that the duty they were called on to perform was rather to carry out a system predetermined by the hasty voice of public opinion, than to devise a plan entirely agreeable to their own views.

It would be beyond my province as well as my ability to describe the vast works which are now being attempted in conformity with the last resolution of the Metropolitan Board. It will however be sufficient to describe the general principle, in order to enable us to decide how far the two great objects, which any reasonable person must place before him, will be met. These are first, and I say especially, as it to a great extent includes the second, the economical use of sewage; second, the beauty and healthfulness of the metropolis. And here it is most deeply to be regretted that the projectors generally, instead of applying themselves to the fair consideration of both the above objects, have hastily abandoned the first one, so that, even if the plans answer the intention of the designers, the first great object will be further than ever from its realization. In fact, to illustrate how steadily, and I may say determinedly, the opposition to economy has been carried on, I have only to quote the following language of the government referees in page 33 of their *Report*: — "We consider it very inexpedient for the Metropolitan Board of Works to adopt any plan which is based upon the deodorization or the utilization of sewage; that if an attempt is to be made to utilize London sewage, it should be made by private enterprize;" and in page 43 — "That the value of the fertilizing matter contained in London sewage is undoubtedly great; but that the large quantity of water with which it is diluted precludes the possibility of separating more than about

one seventh part of this fertilizing matter by any known economical process; that a copious dilution of the sewage is necessary to the health of the inhabitants of the metropolis; and that therefore the sacrifice entailed by the dilution must be endured." — The plain meaning of all this I take to be: *We will take care to dilute and remove the sewage, and then when, as we have shown, private enterprise will be unremunerative, we will invite it.*

Sketch of the scheme of the Metropolitan Board. — On the north side the scheme consists of a main high-level sewer, to intercept the fall from the higher parts, extending from Hampstead; a main middle-level sewer from Kensal Green; and a main low-level sewer from Vauxhall Bridge Road. All these terminate near Bow, whence the united streams pass in a channel formed of a triple culvert of brickwork to Barking Creek. On the south side a similar system of high-level and low-level sewers is to extend from Clapham and Putney to Greenwich, and thence to be carried forward in one main through Woolwich to Crossness Point, a place midway between Woolwich and Erith. Pumps are to be employed to raise the sewage at certain points, and storm-overflows are to enable the mains to discharge themselves through the previous system of sewers into the river within the limits of the metropolis, whenever in consequence of a sudden fall of rain the former are overcharged. Two large reservoirs are to be placed at the outfalls of the two great mains, with the object of retaining the sewage until after full tide, when it is to be discharged into the Thames.

The utilization of sewage is virtually ignored in the scheme of which I have just given an outline. Will it answer the object for which it is solely designed — that of purifying the Thames, and increasing the healthfulness of the district? To reply to this question, we must consider 1st the operation of the principle of intercepting and di-

verting the sewage from its original course. The present sewers in their usual functions will have to be considered as taking their rise at the points where they are crossed by the mains. Hence their size will be larger than it ought to be for the diminished current, and accumulations will result, which latter will be carried in time of storm-overflow into the Thames. A striking proof that such accumulations are, even under the present system, liable to take place and be carried off during storms is adduced by Dr. Hofmann and Mr. Witt. These chemists state that when after a sudden and heavy fall of rain the flow of the Savoy Street sewer had increased sixfold, they found that instead of the sewage being thereby diluted, a given volume actually contained more than twice the quantity of solid constituents which it contained under normal circumstances.

2nd. I doubt whether mains built of brick, however well cemented, can be depended upon to convey sewage. Brick is usually porous, and in that state it cannot be doubted that sewage-water will filtrate through it and thus gradually contaminate the adjacent ground. The injurious effects of such infiltration ought not to be overlooked in a system of mains extending to a total length of sixty miles.

3rd. That portion of sewage which arrives at the outfalls will not be entirely prevented from returning to the metropolis. I arrive at this conclusion from the fact that the sea-water penetrates occasionally as far as London Bridge. The river is frequently brackish at Barking Creek and Woolwich. Experiments with floats may induce fallacious conclusions in this respect, since it is probable that the scour of the flood-tide at the bottom of the estuary is greater than that of the ebb-tide.

4th. The Thames will be rendered particularly noxious at the point where so vast a quantity of offensive matter is to be concentrated. By what justice a nuisance

can be removed from ourselves to be placed under the noses of our neighbours I know not. Nor can I appreciate the wisdom of sacrificing the purity of the air inhaled by the inhabitants of Greenwich, Woolwich, Gravesend, &c., and the immense floating population,* in the doubtful attempt to make the air of the metropolis more wholesome.

5th. The air confined in the new drains will be a serious increase to the already enormous volume of putrid gases in the sewers. The government Referees make just and forcible remarks upon this evil. They state that "the effect of trapping the street gully-drains, without providing other ventilation of the sewers, is, that the noxious gases generated in the sewers are forced into the houses when the flow of sewage increases, the syphon-traps of water-closets and sinks being the points at which the least resistance is presented to their escape from the sewers. To obviate these evils, the plan has been partially adopted of providing in the middle of the street untrapped openings into the sewers. These openings must be endured until a better mode of ventilation shall be adopted, although the foul smells they emit are frequently very great nuisances."† In addition to the cause assigned by the Referees for the expulsion of the poisonous gases, I will mention the changes of atmospheric pressure. The fall of the barometer of one inch will of course occasion the liberation of one-thirtieth of the entire volume of gas. The smell so generally observed to arise in the neighbourhood of drains before rain may probably be referred to this cause. It is also worthy of remark, that in winter the comparatively warm air of the sewers will have a tendency to rise. May not the greater mortality during that part of the year be partly attributed to this circumstance?

* "It is extremely undesirable in a sanitary point of view to cause sewage-water to be intermixed with sea-water." — Messrs. Hawksley, Bidder and Bazalgette's *Report*, p. 52.

† *Report of Government Referees*, p. 40.

6th. The proposed system must be considered a filthy one, as instead of removing sewage to the soil, which is the natural deodorizer, it will cause its accumulation in the bed of the river at a distance of only a few miles from the city. Even the liquid portion will remain for months near the spot where it is introduced, as is proved by the experiments of Mr. Forster, who found that a float put into the river at Barking advances only five miles in its course towards the sea in an entire fortnight.*

The above are some of my reasons for believing that the proposed plan of the Board of Works will fail in promoting the object to attain which the promoters have sacrificed what ought to have been their first consideration. I enter not now on various points, such as the destruction of fish in the river and in the wells of ships, and consequent interference with a useful trade; the formation of banks apprehended by some, and the consequent impediment to navigation; the expense; and other details which must be of minor importance in a question ultimately involving the life and subsistence of an entire population.

The government Referees remark that the pollution of streams by sewage throughout the country is an evil which is increasing with improved house-drainage,† and they sought to place the outfalls as low as Sea Reach. The present is a plan which supplements and perpetuates the

* *Report of Government Referees*, p. 172. Walter Crum Esq., F.R.S., has suggested to me as very possible that there may be times when greater quantity of water enters the Thames by flood than goes out by ebb-tide, owing to the large quantity of water taken from the river by evaporation in dry and hot weather. — See Dalton on Rain, Evaporation, &c., *Manch. Memoirs*, vol. v. p. 346.

† The pollution of springs is a still more serious evil, in many instances involving the necessity of conveying the rain which falls on the moors through a long series of pipes. It is to be doubted whether such water is as good for drinking purposes as uncontaminated spring water.

evil of which they complain; a new patch is to be added to the old garment, and as a natural consequence the rent will be made worse. An evil ought to be honestly and fairly met, not merely slurred over and disguised.

I now pass from the consideration of works which when completed will, there is every reason to believe, result in total failure, and will endeavour to show the practicability of realizing the first great object of removing sewage, viz., its utilization. Much has been said on this subject since Mr. John Martin in 1828 drew attention to the waste which was even then going on; but judging from the acts of public bodies, it would seem to be still doubtful in the minds of a great portion of the community whether the saving of the manure of cities is a matter of any considerable importance. It is therefore desirable that the actual facts should be constantly brought under review. For this purpose I might bring the evidence of nearly every scientific chemist, but will content myself with quoting from Liebig, both on account of the great attention he has paid to the subject and the circumstance that, having been interested in artificial manures, he is not liable to be unduly biassed in favour of natural ones. At page 177 of his *Chemistry of Agriculture and Physiology* this distinguished philosopher says: "The mineral ingredients of food have been obtained from our fields, having been removed from them in the form of seeds, of roots, and of herbs. In the vital processes of animals the combustible elements of the food are converted into compounds of oxygen, while the urine and fœces contain the constituents of the soil abstracted from our fields; so that by incorporating these excrements with our land we restore it to its original state of fertility. If they are given to a field deficient in ingredients necessary for the growth of plants, it will be rendered fertile for all kinds of crops. A part of a crop taken from a field is used in feeding and fat-

tening animals, which are afterwards consumed by man. Another part is used directly in the form of potatoes, meal, or vegetables; while a third part, consisting of the remnants of plants, is employed as litter in the form of straw, &c. It is evident that all the constituents of the fields, removed from it in the form of animals, corn and fruit, may again be obtained in the liquid and solid excrements of man, and in the bones and blood of slaughtered animals. It altogether depends upon us to keep our fields in a constant state of composition and fertility by the careful collection of these substances. We are able to calculate how much of the ingredients of the soil are removed by a sheep, by an ox, or in the milk of a cow, or how much we convey from it in a bushel of barley, wheat, or potatoes. From the known composition of the excrements of man, we are also able to calculate how much of them it is necessary to supply to a field to compensate for the loss that it has sustained." Again, in page 181, he says: "In the solid and liquid excrements of man and of animals, we restore to our fields the ashes of the plants which served to nourish these animals. These ashes consist of certain soluble salts and insoluble earths which a fertile soil must yield, for they are indispensable to the growth of cultivated plants. It cannot admit of a doubt that, by introducing these excrements to the soil, we give to it the power of affording food to a new crop, or, in other words, we reinstate the equilibrium which had been disturbed. Now that we know that the constituents of the food pass over into the urine and excrements of the animal fed upon it, we can with great ease determine the different value of various kinds of manure. *The solid and liquid excrements of an animal are of the highest value, as manure for those plants which furnished food to the animal.*"

From the above incontestable principles we may easily calculate the magnitude of the loss which is sustained by

the waste of sewage. If the excrements of an animal are not returned to the soil the food of that animal cannot be reproduced. Hence the amount of barrenness communicated to the soil by the system now endeavoured to be enforced in our towns may, considering no food or manure to be imported from other countries, be directly estimated by the food consumed by the inhabitants. Dalton, in the fifth volume of the *Manchester Memoirs*, 2nd series, has given the aggregate of the articles of food consumed by himself in fourteen days, his habits, daily occupations, and manner of living being exceedingly regular. They are —

Bread	163 oz.	Milk...	435½ oz.
Oatcake.....	79 „	Beer...	230 „
Oatmeal	12 „	Tea ...	76 „
Butcher's meat...	54½ „		
Potatoes	130 „		
Pastry	55 „		
Cheese	32 „		
Total.....	525½ „ solids.		741½ „ fluids.

Much more than the above quantities are consumed by the luxurious, much less by the aged and invalid. I think on the whole, and for our present purpose, that we may take them as the food of every man, woman and child in the metropolis. Hence we may infer that the 2,600,000 inhabitants of London consume every day provisions equivalent to —

1,316 tons of bread.	505,000 gallons of milk.
282 tons of butcher's meat.	267,000 gallons of beer.
674 tons of potatoes.	88,000 gallons of tea.
285 tons of pastry.	
166 tons of cheese.	
Total...2,723 tons of solid, and	860,000 gallons of liquid food.

This is therefore the daily rate at which the productive power of the country suffers by the waste of one large town, and this is done in the face of a rapidly increasing

population. Yet there are many who treat the subject entirely as a commercial one, and if the cost of transit is such as to prevent sewage competing with guano in the market, they argue that it ought to be thrown away as refuse. But this is a fallacious, narrow-minded and selfish view of the subject. In order apparently to save ourselves a little money at the present moment it entails a heavy burden on the inhabitants of the country in subsequent years. Guano will not last for ever. According to the Peruvian Survey the Chincha Islands can yield 18,200,000 tons. Of this quantity Great Britain alone consumed in 1857 no less than 288,362 tons, which, if we consider the entire waste of sewage in Great Britain to be double that of London, will almost exactly make up for it in money value. If the produce of the above islands, which afford the best guano, be reserved for the sole use of Great Britain, it will last only sixty-three years at the present rate of consumption. I am aware that other supplies have been found in various parts of the world, and that there is a trifling additional deposit each year. But when we consider the competition which will eventually take place on the part of other countries to secure so valuable a manure, and also the ever increasing difficulties of obtaining it, we cannot trust to our being able to import it in the quantities we now do for even so long a period as that above named.

In addition to the help derived from guano, the soil of Britain is relieved from the present effects of sewage waste by large importations of corn and cattle and of bone manure. But such a supply can continue only so long as foreign governments remain in ignorance of the permanent injury sustained by their fields. Liebig, in his *Agricultural Chemistry*, complains that, if the exportation of bones continued on the then scale, the German soil would become gradually exhausted.

Besides, we ought not to be satisfied with merely keeping the productive power of our agriculture from decline. With a rapidly increasing population the wisest course would be to reserve such supplies of guano as we may be able to obtain for the purpose for which nature appears to have designed it, that of forming a fertile soil where sterility at present exists.

In concluding this part of my subject I would urge the importance of recollecting that in the estimate of fertility, regard should be paid not only to the weight of a crop but also to its nutritive value, determined in each case by chemical analysis. Liebig states that an increase of animal manure gives rise not only to an increase of the number of seeds, but also to a most remarkable increase in the proportion of those nitrogenous substances which are the most important constituents of food.

Having endeavoured to show the imperative necessity, I will say a few words on the means, of putting a stop to the present waste. The first step I conceive should be to prohibit the introduction into the sewage of any organic matter which can be avoided. For instance, scavengers should be constantly employed in collecting and removing horse dung from the streets. The present system of sending carts round at long intervals of time allows by far the larger portion of this manure to be washed by rain into the sewers, thus forming a very serious addition to their impurity.

Then why should slaughter-houses be tolerated? If only meat slaughtered in the country were admitted into the town, I submit that the meat would be cheaper in regard to its intrinsic nutritive value. The distress suffered by the animals in their passage from the field to the town slaughter-house destroys the richness and flavour of their meat, even if it do not render it positively unwhole-

some. By the present system a large quantity of offal and blood is removed from the country, where it would be a valuable manure, to the town, where it is a dangerous nuisance.

I may mention in this place the subject of intramural interments, which even at the present day have not been entirely discontinued. Tens of thousands of human bodies in a disintegrated and decomposed state have floated down the sewers of London into the Thames. The drainage of burial grounds into sewers is, in fact, enjoined by act of parliament. Now the body of any human being after death ought, in accordance with the Divine ordinance, to be permitted to return to the dust whence it came. For this purpose metallic coffins are unsuitable; and the body should be placed at a moderate depth below a soil on which there is a vegetable growth. I cannot enter into details on this highly important subject, but I am satisfied that the object of rapid conversion into vegetable life may be attained without in the least degree hurting, but rather subserving, those feelings of affection and reverence with which we regard the dead.

After prohibiting the unnecessary introduction of organic matter, the next step will be to deal with the sewage proper. And here we find at the outset that the enormous quantity of water mixed with it in the drains prevents the possibility of using it in that state for agricultural purposes. Messrs. Bidder, Hawksley and Bazalgette, among other objections, come to the following conclusions in their Report. *First:* "That the fertilizing properties of the organic matters contained in town refuse are for the most part destroyed by the long continued action of water." *Second:* "That the cost and difficulties attending the application of liquid sewage in large quantities are absolutely prohibitory of its use." *Third:* "That liquid sewage cannot in general be used with advantage in this climate,

except in particular states of the weather, and in certain stages of the growth of the crops to which it is applied."* The precipitation processes by lime, &c., even though at present commercially valueless, ought to be persisted in, if it is only in our power to deal with largely diluted sewage. But, according to Hofmann and Witt, not more than one-third of the fertilizing constituents can be thus separated. It is obvious, therefore, that we should deal with the sewage in a more concentrated form and before it is diluted with rain and other comparatively clear water. The separate system has been frequently advocated, but there is some doubt whether sewage in a concentrated form would flow through a long series of pipes of very moderate inclination. With these facts before me I see no alternative but a return to the cesspool system, to which I believe no inconvenience or nuisance attaches, except where it is attempted carelessly and with inefficient mechanical and other appliances. The following is a plan which I venture to recommend in places where water-closets are generally used.

I would place in the centre of the streets, cesspools having a capacity of about 1000 gallons. Each cesspool to be for the use of some 400 inhabitants, say 50 houses, and to collect water from urinals, cab stands, &c. The present sewers to be solely employed in carrying off rain and other comparatively clear water.

A drain, or generally two drains of considerable inclination extending from the cesspool up and down the street, to receive the water-closet pipes from the houses on both sides of the street. The total length of drain would be about 200 yards.

A force-pump, permanently fixed in the cesspool, to be used every night for the purpose of pumping out the sewage collected in it during the last 24 hours.

* *Report*, p. 104.

The sewage thus pumped to be discharged into tanks, and then conveyed to a railway to be carried to reservoirs situated at convenient localities in the country.

Each tank might have a capacity of ten tons, and would then hold the contents of five cesspools. It might be drawn by a traction steam-engine, which also might be employed for the pumping. The discharge pipe of the force-pump, as well as its piston-rod, might rise to the level of the street, and the requisite connexions be screwed or clamped on when required. Immediately after emptying the cesspool a portion of McDougall and Smith's disinfecting powder* might be thrown in. This, acting on the sewage at an early period, would, Dr. Smith states, have the best effect in deodorizing, and in preserving the fertilizing property.

I believe that in the above way the sewage of London might be conveyed to the fields, and a very large annual profit realized, instead of the dead loss of three millions sterling, which must be incurred if the plan of the Metropolitan Board is carried out. The other advantages would consist in,— 1st: Easy construction and repair. 2nd: Total prevention of infiltration of sewage, and the effects of accumulations of noxious gas. 3rd: Rapid removal of sewage before decomposition has had time to take place. 4th: An unpolluted river.

* Sulphite of magnesia and lime, and carbolate of lime.

XII. — *An Account of the Fall of Rain at Manchester, from the Year* 1786 *to* 1857 *inclusive.*

By Mr. John Curtis.

Read November 16th, 1858.

This paper has been prepared with a view of making the tables complete to the end of last year, and of showing at a glance the amount of rain which has fallen during the last 72 years. The account of the fall of rain from 1786 to 1793 inclusive is taken from the tables prepared by Mr. George Walker, and published in the Society's Memoirs, vol. iv. pp. 584 and 585, old series. From 1794 to 1840 inclusive, from the tables prepared by Dr. Dalton, and published in the Society's Memoirs, vol. v. part ii. p. 668, old series; vol. iii. p. 496, and vol. vi. pp. 575 and 576, new series. From 1841 to 1854 inclusive, from observations taken by Mr. Joseph Casartelli, of this city, who kindly communicated them to me. And from 1855 to 1857 inclusive, from observations taken by myself in Plymouth Grove, Chorlton-upon-Medlock, on the south side of the town. The gauge I employ is a funnel 8½ inches in diameter, with a perpendicular rim 5 inches high, and the top of the rim is 2 feet 3 inches above the ground, the fall of rain being registered daily at ten p.m. in a graduated glass cylinder. The gauge is situated in a garden perfectly free from surrounding objects, the nearest buildings being on the south side, and twenty to thirty yards distant; while it is more than thrice the above distance on the other sides from elevated objects. Dr. Dalton's gauge was a funnel 10 inches in diameter, surrounded by a perpendicular rim 3 inches high, the top of which was a little more than two

feet from the ground, and was situated in a garden on the south-east side of the town, and twenty yards distant from any house or elevated object.* Mr. Casartelli's gauge was a funnel 5 inches in diameter, with a perpendicular rim 2 inches high, the top of which was between two and three feet above the ground, and was situated on the south-east side of the town, ten yards distant from a house on two sides, and the same from a wall nine feet high on the two other sides. It will thus be seen that the gauges employed in registering these observations were all of the same construction, with very little difference in their distance from the ground, and that the places where they were registered were within one mile from each other. They are also of the kind now recommended by Mr. Glaisher, of the British Meteorological Society, as the best for taking observations on the fall of rain.

It may be well here to state that Dr. Dalton found Mr. Walker's returns to exceed his own by about four inches in the year, and that on inspecting Mr. Walker's gauge he had reason to think that the method of measuring the rain employed was not susceptible of sufficient accuracy, and on his suggesting the same to Mr. Walker, the latter seemed to acquiesce.† For this reason I have given the fall of rain as collected by Mr. Walker in a separate table, so as to enable me to give two averages, the one with his observations included and the other without.

In Dr. Dalton's observations the fall of rain in the months of March and April 1807, December 1809, and January and February 1810, are not given. To fill up these blanks, and make the tables complete, I have inserted the average fall of rain in those months, so that I am enabled to give the mean and total for each month and year, and also the mean and total for each month and year during the entire series of seventy-two years.

* See vol. iii. p. 195, second series, of the Society's Memoirs.

† See Dr. Dalton's remarks, vol. iii. p. 498, new series.

From the foregoing tables it will be seen that, including Mr. Walker's returns, the average fall of rain during the 72 years is 36·3988; and, excluding his returns, the average for the remaining 64 years is 35·5620. Dr. Dalton's average of 47 years is 35·523, showing the small difference of ·039 between the two latter averages. I think we may, therefore, assume that Dr. Dalton was right in supposing that Mr. Walker's returns were in excess of the reality; and, for the purpose of arriving at a correct average, it will be safer to omit them and adopt 35·5620 as the average fall of rain for Manchester.

As the situation of the rain-gauge employed, and the influence of surrounding objects on it will produce different results, the following table has been prepared to show the mean monthly and yearly fall of rain, as registered by each observer, of the whole period and of the 64 years, the average obtained during the latter period being that which I recommend being adopted as nearest the truth.

	Walker. 8 years. 1786 to 1793 inclusive.	Dalton. 47 years. 1794 to 1840 inclusive.	Casartelli. 14 years. 1841 to 1854 inclusive.	Curtis. 3 years. 1855 to 1857 inclusive.	Total 72 years. 1786 to 1857 inclusive.	64 years. 1794 to 1857 inclusive.
	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.
January .	2·4685	2·257	3·335	2·150	2·4890	2·4915
February	2·7499	2·443	2·437	1·946	2·4556	2·4190
March . .	2·1145	2 308	2·261	1·526	2·2427	2·2588
April . . .	2·3019	2·114	1·697	2·123	2·0536	2·0225
May. . . .	3·5103	2·446	2·134	2·376	2·5008	2·3746
June . . .	3·3000	2·691	3·319	3·106	2·8985	2·8483
July . . .	4·5832	3·706	3·793	3·663	3·8187	3·7231
August .	4·7499	3·463	3·767	4·356	3·7025	3·5715
September	4·2144	3·192	3·103	2·356	3·2552	3·1353
October .	4·5104	3·754	4·262	3·223	3·9150	3·8404
November	3·3019	3·712	3·416	2·020	3·5386	3·5682
December	5·2873	3·487	3·148	2·012	3·5286	3·3088
Total . . .	43·0922	35·523	36·672	30·857	36·3988	35·5620
First 6 months	16·4451	14·259	15·183	13·227	14·6402	14·4147
Second 6 mos.	26·6471	21·264	21·489	17·630	21·7586	21·1473

The preceding table shows that, taking the average, April is the driest and October the wettest month in the

year, and that the fall of rain in the first 6 months of the year is to that of the last 6 months as 2 to 3 — that there is less difference between the two periods in the later returns than in the earlier ones; for Mr. Walker's returns show a proportion of 16 to 26, Dr. Dalton's of 14 to 21, Mr. Casartelli's of 15·1 to 21·24, and my own of 13·2 to 17·6. Whether this progressive diminution of difference between the first and the last half of the year is owing to the later returns being for shorter periods than the former ones, or to some change in the fall of rain influenced by a change of circumstances, future returns will show.

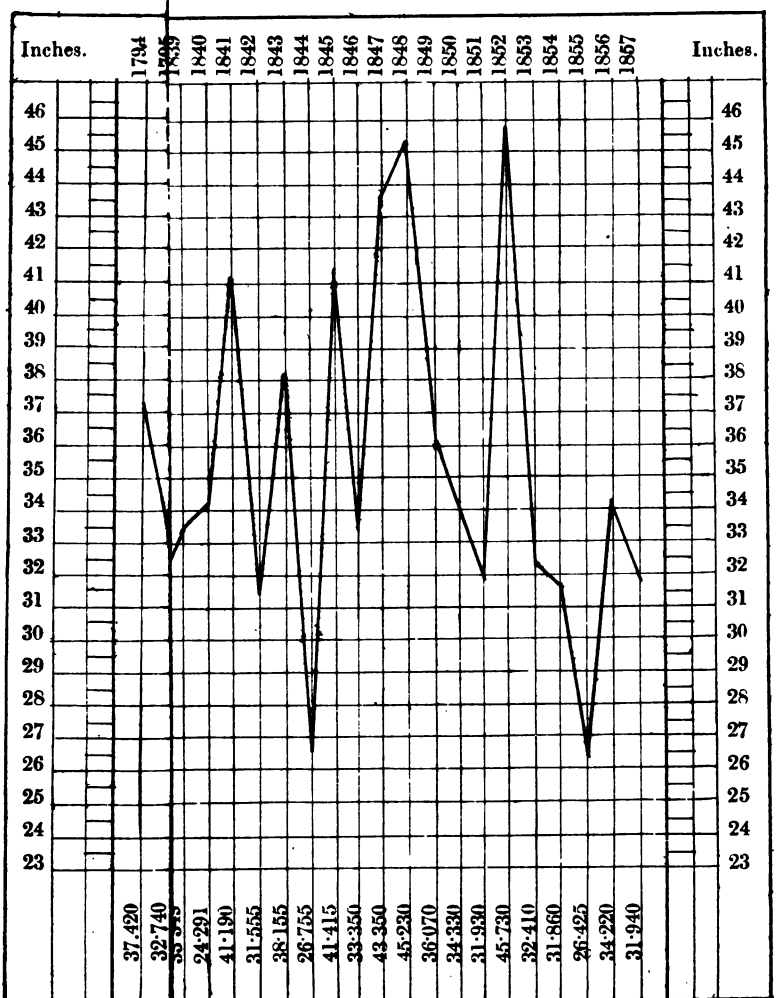
The following table shows the greatest and least amount of rain which has fallen in every month during the 72 years, with the year in which it fell, and the name of the collector.

	Year.	Inches.		Year.	Inches.	
January .	1806	5·851	Dalton.	1833 1838	0·320	Dalton.
February	1848	6·565	Casartelli.	1800	0·440	Dalton.
March . .	1827	6·030	Dalton.	1808 1829	0·180	Dalton.
April . . .	1791	4·750	Walker.	1842	0·160	Casartelli.
May . . .	1792	8·000	Walker.	1844	0·090	Casartelli.
June . . .	1830	7·055	Dalton.	1826	0·200	Dalton.
July . . .	1828	11·480	Dalton.	1800	0·290	Dalton.
August .	1799	8·740	Dalton.	1801	0·730	Dalton.
September	1792	9·000	Walker.	1804	0·240	Dalton.
October .	1787	9·000	Walker.	1817	0·604	Dalton.
November	1825	7·375	Dalton.	1805	0·624	Dalton.
December	1792	9·500	Walker.	1844	0·070	Casartelli.

The largest amount of rain fell in 1792, and was 55·250 inches; the least amount of rain fell in 1826, and was 24·910 inches.

The rain which fell from 1793 to 1814 was below the average of the 64 years; from 1815 to 1852 it was above; and from 1853 to the present time it has again fallen below, leading to the inference that we have entered into a low series, and that, consequently, we may for some time expect the rain fall to remain below the average, though

ard inch, by Mr. John Curtis, from the data of
clusive; and of Mr. John Curtis



Fall of Rain at Manchester from 1786 *to* 1793 *inclusive, as collected by Mr. Walker; with the monthly and yearly fall and the monthly and yearly mean average of the same.*

Year.	January. Inches.	February. Inches.	March. Inches.	April. Inches.	May. Inches.	June. Inches.	July. Inches.	August, Inches.	September. Inches.	October. Inches.	November, Inches.	December. Inches.	Total Yearly Fall. Inches.
1786	2·500	1·500	2·166	0·916	3·500	4·000	2·666	6·500	7·833	2·750	1·416	4·666	40·413
1787	1·000	3·500	3·000	1·750	2·750	2·500	7·666	3·666	2·166	9·000	5·666	4·833	47·497
1788	1·750	1·666	2·500	1·833	1·833	1·000	4·833	3·750	2·000	2·833	2·333	1·000	27·331
1789	2·000	5·333	1·500	2·666	4·500	6·500	7·250	0·833	4·250	5·750	4·750	5·666	50·998
1790	2·250	1·250	1·000	2·250	3·250	5·500	5·750	4·500	3·750	2 750	3·250	7·250	42·750
1791	5·500	3·000	1·500	4·750	2·750	0·750	3·500	6·000	1·416	4·500	4·500	5·833	43·999
1792	2·000	2·000	2·750	2·500	8·000	3·500	3·750	6·250	9·000	4·000	2·000	9·500	55·250
1793	2·750	3·750	2·500	1·750	1·500	2·650	1·250	6·500	3·300	4·500	2·500	3·550	36·500
Totals.	19·750	21·999	16·916	18·415	28·083	26·400	36·665	37·999	33·715	36·083	26·415	42·298	344·738
Means.	2·4685	2·7499	2·1145	2·3019	3·5103	3·3000	4·5832	4·7499	4·2144	4·5104	3·3019	5·2873	43·0922

Fall of Rain at Manchester from 1794 *to* 1857 *inclusive, as collected from* 1794 *to* 1840 *by Dr. Dalton, from* 1841 *to* 1854 *by Mr. Casartelli, and from* 1855 *to* 1857 *by Mr. John Curtis; with the monthly and yearly fall and the monthly and yearly mean average of the whole.*

Year.	January. Inches.	February. Inches.	March. Inches.	April. Inches.	May. Inches.	June. Inches.	July. Inches.	August. Inches.	September. Inches.	October. Inches.	November. Inches.	December. Inches.	Total Yearly Fall. Inches.
1794	1·090	3·770	2·030	3·910	2·270	1·250	3·400	4·380	4·660	5·100	3·450	2·110	37·420
1795	0·950	2·460	2·020	3·230	1·010	2·060	3·010	4·280	0·460	5·490	4·160	3·610	32·740
1796	3·100	2·290	0·580	1·600	5·090	2·350	5·190	1·040	2·630	2·870	2·130	1·760	30·630
1797	1·580	1·210	0·940	2·180	5·960	4·260	2·520	5·490	3·860	2·310	2·990	5·500	38·800
1798	2·700	1·910	1·180	1·370	1·480	0·850	4·750	4·130	3·350	3·510	3·680	2·350	31·260
	1·760	4·160	2·160	2·190	2·130	2·050	4·080	8·740	5·350	3·770	1·900	0·350	38·640
	·860	0·440	2·370	4·100	2·850	0·710	0·290	1·050	6·550	4·370	3·700	3·050	32·340
	·420	1·850	2·890	0·790	2·600	0·530	4·850	0·730	6·410	4·270	3·150	4·560	35·050
	·967	3·728	0·860	1·972	0·860	3·120	6·233	3·221	2·176	6·126	1·882	3·592	35·737

will doubtless be some years among the series in h it will be above the average. A reference to the t will show the variation of the series very clearly, by reference to the tables the following remarkable erences will be found. From 1795 to 1814 inclusive mean of 20 years is 33·044 inches; from 1815 to 36 inclusive the mean of 22 years is 38·161; from 37 to 1852 inclusive the mean of 16 years is 36·328; hile from 1853 to 1857 inclusive the mean of 5 years s only 31·371; showing the correctness of Dr. Dalton's emarks as to the importance of a long continued series of observations to obtain a satisfactory table of the mean quantity either for each month or the whole year.

XIII. — *Proposed Improvements in Pharology.*

By Richard Roberts, M. Inst. C. E.

Read March 8th, 1859.

The well-authenticated fact that the great majority of wrecks and collisions occur in the immediate vicinity of the lighthouses intended to guard against them, has caused of late many schemes to be propounded for the improvement of our present system of Pharology. Amongst these, that of Mr. Herbert, of the Admiralty, is, I think, the most deserving of attention, as there can be no doubt that most of these disasters have been occasioned by the universal practice of sailing along the coasts within sight of the lights and taking a fresh course when the general trend of the coast changes its direction; thus involving the necessity of nice calculation or great knowledge of the coast, often unavailing through ignorance or want of care on the part of those navigating in an opposite direction.

Mr. Herbert proposes to moor in a direct line up the English and other Channels a series of floating lighthouses (if I may use the term) of such character and power of illumination that they shall no longer be rivalled by the lights carried by steamers, which are now often mistaken for them. With this suggestion I most cordially agree, and I think that were such a line of lights once established, making it imperative that all vessels leave them on the port hand, we should see in a future record of collision and wreck a far smaller list of calamities than we now find crowded around our lighthouses. My attention was first called to this subject by a paper read by a Mr. Murphy in

section G of the British Association, at its meeting recently held at Leeds, and it at once occurred to me that the principle of gyration might be advantageously applied to neutralize the powerful action of the wind and waves upon floating light-beacons. I have since then given some attention to the subject, and have embodied my ideas in the diagram which I have now the honour to submit to the Society.

Before describing my proposed improvements it will be necessary, for the sake of perspicuity, briefly to consider the two systems now in use for the illumination of light-houses, viz: the *catoptric* or English system, which consists in the use of *many* lamps with *metallic reflectors* placed *behind* the lights, and the *dioptric* or lenticular system, which consists of a *number of lenses* united or built up so as to form a vertical octagonal hollow prism, which, circulating around a *single* light fixed in the centre, shows to a distant observer successive flashes or blazes of light whenever one of its faces crosses a line joining his eye and the lamp. The invention of this system is, I believe, due to the late Mons. Augustin Fresnel, whose name it bears.

The object to be attained by the use of both these systems is of course the same, namely, to collect the rays of light which diverge from a point called the *focus*, and to project them forward in a beam whose axis coincides with the produced axis of the instrument; but the means whereby they attain this end are different, this result being produced in the catoptric system by the light being *reflected* or *thrown back* from a surface so formed as to cause all the rays to proceed in one and the same direction; whilst in the dioptric system they pass *through the refracting medium* and are *bent* or *refracted* from their natural course into that which it is desired they should take. It is impossible by any combination of paraboloidal reflectors to distribute around the horizon a zone of light of equal

intensity, while the dioptric method *perfectly* fulfils this condition by distributing the rays equally to every point of the horizon; and not only does the French apparatus, as lately improved, produce as the average effect of the same combustion of oil about four times the amount of light that is obtained from the catoptric mode, but its annual maintenance, including interest on first cost of apparatus, is considerably less. In the event, however, of the whole horizon not requiring to be illuminated, the dioptric light would be more expensive than the reflected light; but the greater power and more equal distribution of light may be considered of such great importance as to outweigh the difference of expense. On the other hand, the catoptric system insures a more perfect exhibition of the light, not only from the fountain lamps being less liable to derangement than the mechanical lamps used in dioptric lights, but because the extinction of one lamp in a catoptric light leads to much less serious consequences than the extinction of the single lamp in a dioptric light. Experience, however, goes far to show that in practice the risk of extinction of the lamp in dioptric lights is very small, and I think that there can be little doubt that the more fully the system of Fresnel is understood, the more certainly will it be preferred to the catoptric system for the illumination of shore lighthouses. It must, however, be evident that the great oscillation to which lightships as at present constructed are subject, renders all arrangements of glasses for the dispersion of light in a zone inapplicable to them. Our lightships are therefore still furnished with twelve-inch reflectors (mounted on gimbols), which, by their great divergence of the light, obviate the objections to which the dioptric system would be subject; although that divergence involves a corresponding loss of intensity.

The conditions required for a floating light to be effi-

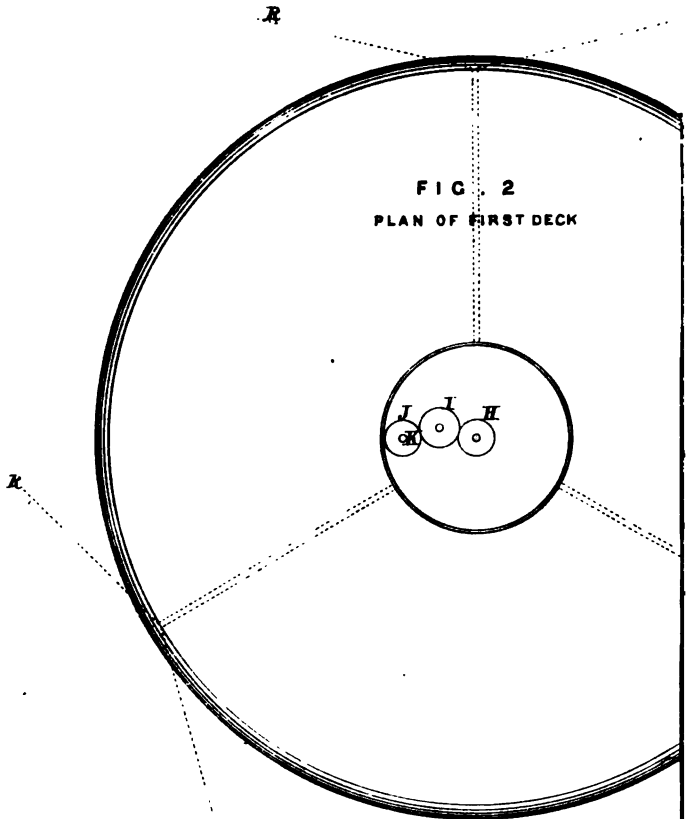


FIG. 2

PLAN OF FIRST DECK

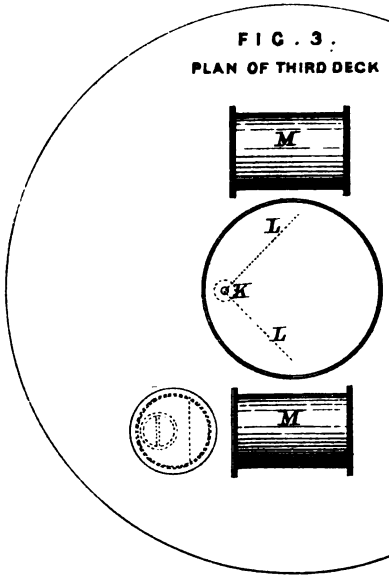


FIG. 3.

PLAN OF THIRD DECK

cient are, that it should keep upright and be free from any oscillation such as is experienced by ordinary vessels. The means whereby I propose to attain this desideratum will, I think, be readily understood on inspection of the accompanying plate, in which all details are purposely omitted to avoid confusion. I propose entirely to change the form of the vessel, making that portion of it which is immersed hemispherical, and that which is above water the frustrum of an inverted cone; as I think that this form of float will present less resistance than any other to the action of the wind and waves where these are expected to act from every point; although it may be advisable in certain situations to employ a vessel pointed to both ends. Through the centre of the float I propose to pass a cylindrical tube B (*Fig.* 1), the lower end of which shall project through the bottom of the float so as to form a hollow keel for the reception of ballast. That portion of the tube which is above the deck of the float serves as a tower upon which to mount the light, and may be divided into rooms for the accommodation of the light-keepers, &c. C is the lantern, similar to those of shore lighthouses, and D the gallery surrounding the lantern. I propose to employ a dioptric light apparatus of the second order, shown at EE, forty-five feet above the water, which will permit the light to be seen at a distance of nine miles. F is a fly-wheel, or gyrator, mounted on gimbols placed a little above the centre of gravity of the light apparatus and fly-wheel taken together. This fly-wheel is placed on the upper end of the shaft G, which receives motion through the wheels H, I and J and the shaft K, from an engine or pair of engines placed on the third deck of the float; or the fly-wheel may be kept in motion at its proper speed by two relays, each of three men.

MM (*Fig.* 3) are small high-pressure steam boilers for

working the engine, which, in addition to turning the fly-wheel, may be employed to pump water, hoist coals and stores aboard, sound bells, &c.; and I think that steam might be advantageously used to keep the light room sufficiently warm to prevent the adhesion of snow to the windows, to warm the barrack rooms, and to sound one or more whistles.

nn (*Fig.* 1) is the casing surrounding the fly-wheel, and *o* a bridge piece bolted to it, concentric with which it carries a stud upon which is the plate that maintains in its position the light apparatus, whose weight is borne by a number of rollers which are carried round with the fly-wheel.

p is the flame of the lamp, whose pump is to be worked by the shaft K, *q* the glass chimney, q^1 a metal tube fixed to the frame which carries the lenses, and q^2 a telescopic metal tube, the lower end of which is carried by gimbols at the top of the frame carrying the lenses, and supported at its upper end by gimbols in the top of the dome. This tube works freely in the exit tube q^3. I propose that this lighthouse be moored by three anchors, and that to each of these two heavy cables be attached, which shall pass through hawse pipes in the side of the float situate 120° from each other, as shown in the small drawing. I also propose to employ a suitable windlass, so adapted as to give the requisite tension to the six chains. By this mode of mooring, the vessel will be always kept over one particular spot, and the cables will surge much less; there will consequently be less risk of the anchors dragging, whilst from whatever quarter the strain may come the resistance will be about the same. I may add that the central portion of the first deck is raised above the outer portion to prevent water flowing on to it. I also propose to have one or more life boats attached to these floats for the purpose of rescuing the crews of ships in distress, or for

the escape of the crew of the lightvessel in case of accident.

Having thus briefly described my proposed plan, I would observe that I think, that by the gyrations of the fly-wheel (about 100 per minute), the peculiar form of float which I employ, and the manner in which I moor it, the desired object will be attained, that is, the oscillation will be reduced to a minimum, and we shall be able, not only to increase the intensity of the light by the use of the most improved apparatus, but likewise to extend its range by adding to the height of the tower.

It may be objected that the use of an engine would greatly increase the expense of maintenance; this however would not be the case, as a small engine, with fuel, attendants, &c., might be maintained for about £150 per annum, even if no corresponding reduction were made in the number of hands now employed. I may add that the total weight of this floating lighthouse, with all its machinery, stores, &c., on board, will be about 300 tons, whilst its displacement will be equal to 600 tons; it will therefore perhaps be practicable to diminish the amount of immersion, and so add to the height of the tower.

XIV. — *On the Method of Symmetric Products, and its Application to the Finite Algebraic Solution of Equations.*

By the Rev. Robert Harley, F.R.A.S.

Read April 5th, 1859.

The following investigations were suggested by Mr. Cockle's paper, entitled "Researches in the Higher Algebra," which I had the pleasure of communicating to the Society a few months ago. In that paper a certain function of the roots of a quintic is discussed. By an indirect but effective process its author succeeds in obtaining the sextic of which that function is the root. In the present paper Mr. Cockle's results are verified by a strictly independent method, and one which comes commended by its simplicity, directness and generality. Other results, of greater or less importance, are evolved.

Mr. Cockle first explained his method of symmetric products in a series of five papers "On the Transformation of Algebraic Equations," printed in the first and third volumes of *The Mathematician*, a journal of which the first number was published in November 1843. Several papers proceeding from the same pen on the same subject have also appeared in the *Mechanics' Magazine*, the *Cambridge and Dublin Mathematical Journal*, the *Lady's and Gentleman's Diary*, the *Philosophical Magazine*, and in the *Quarterly Journal of Mathematics*. But as this memoir will probably fall into the hands of many who cannot con-

veniently consult those works, I propose to give, in the first section, a brief exposition of Mr. Cockle's method (with alterations), and to indicate its application to the solution of the lower equations.

SECTION I.

The Method of Symmetric Products.

1. Let $x_1, x_2, x_3, \cdots x_n$ be any n symbols, and let X be a linear unsymmetric function of these symbols, such that

$$X = x_1 + a_1 x_2 + a_2 x_3 + \cdots + a_{n-2} x_{n-1} + a_{n-1} x_n,$$

where the $n-1$ constants $a_1, a_2, \cdots a_{n-1}$ are arbitrary. When n is less than 5, these constants may be so distributed and determined as to render the product

$$\pi_{n-1}(x), \text{ or } X_1 X_2 X_3 \cdots X_{n-1},$$

(or, when $n=2$, X^2) symmetric with respect to the symbols $x_1, x_2, x_3, \cdots x_n$. When n is equal to, or greater than 5, the symmetry is in general unattainable; but in seeking to satisfy its conditions we are conducted to significant results. The product, $\pi_{n-1}(x)$, which may be called *the symmetric* or *resolvent product*, according as it is or is not symmetric, plays an important part in the finite algebraic solution of the equation of the n th degree.

2. When $n=2$, we have

$$\{\pi_1(x)\}^2 = X^2 = (x_1 + a_1 x_2)^2;$$

and the condition of symmetry is $a_1^2 = 1$; consequently $a_1 = 1$, or -1; and, since by definition X is unsymmetric, the former of these results must be rejected. Hence $a_1 = -1$, and

$$\{\pi_1(x)\}^2 = (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1 x_2.$$

3. When $n=3$, assume

$$X_1 = x_1 + a_1 x_2 + a_2 x_3,$$

$$X_2 = x_1 + a_2 x_2 + a_1 x_3,$$

and

$$\pi_2(x) = X_1 X_2.$$

Then the conditions of symmetry are

$$a_1 a_2 = 1, \text{ and } a_1^2 + a_2^2 = a_1 + a_2;$$

whence, rejecting those values of a_1 and a_2 which render X symmetric, we have

$$a^2+a+1=0,$$

and a, which denotes a_1 or a_2 indifferently, is obviously an unreal cube root of unity. This root we shall in future designate by ϵ. Let

$$f(\epsilon)=x_1+\epsilon x_2+\epsilon^2 x_3,$$

then $$f(\epsilon^2)=x_1+\epsilon^2 x_2+\epsilon x_3,$$

and $$\pi_2(x)=f(\epsilon)\cdot f(\epsilon^2)=(\Sigma x)^2-3\Sigma x_1 x_2.$$

4. When $n=4$, assume (cyclically)

$$X_1=x_1+a_1 x_2+a_2 x_3+a_3 x_4,$$
$$X_2=x_1+a_2 x_2+a_3 x_3+a_1 x_4,$$
$$X_3=x_1+a_3 x_2+a_1 x_3+a_2 x_4,$$

and $$\pi_3(x)=X_1 X_2 X_3.$$

In this case the conditions of symmetry are

$$1=a_1 a_2 a_3,$$
$$\Sigma a=\Sigma a_1 a_2=a_1^2 a_2+a_1 a_3^2+a_2^2 a_3=a_1^2 a_3+a_1 a_2^2+a_2 a_3^2,$$

and $$\Sigma a^2+\Sigma a_1 a_2=\Sigma a^3+3a_1 a_2 a_3.$$

The first and second indicate that a_1, a_2 and a_3 may be regarded as the roots of an equation of the form

$$a^3-Aa^2+Aa-1=0;$$

and, combining the first four, we have

$$\Sigma a\,\Sigma a_1 a_2=\Sigma a_1^2 a_2+3a_1 a_2 a_3=2\Sigma a+3;$$

or $$A^2=2A+3;\ \text{and}\ A=3,\ \text{or}\ -1.$$

The first value gives

$$(a-1)^3=0,$$

and thus renders X symmetric. We therefore reject it, and, adopting the last, find

$$a^3+a^2-a-1=0.$$

The roots of this cubic are -1, 1 and -1; and all the above equations of condition are satisfied. Let

$$X_1=x_1-x_2+x_3-x_4,$$
$$X_2=x_1+x_2-x_3-x_4,$$

and $$X_3=x_1-x_2-x_3+x_4;$$

then $$\pi_3(x)=X_1 X_2 X_3=(\Sigma x)^3-4\Sigma x\,\Sigma x_1 x_2+8\Sigma x_1 x_2 x_3.$$

5. When $n=5$, assume

$$X_1=x_1+a_1x_2+a_2x_3+a_3x_4+a_4x_5,$$
$$X_2=x_1+a_2x_2+a_4x_3+a_1x_4+a_3x_5,$$
$$X_3=x_1+a_3x_2+a_1x_3+a_4x_4+a_2x_5,$$
$$X_4=x_1+a_4x_2+a_3x_3+a_2x_4+a_1x_5,$$

and $$\pi_4(x)=X_1X_2X_3X_4.$$

Then the condition necessary and sufficient for the symmetry of the terms in x^4 is

$$1=a_1a_2a_3a_4.$$

The corresponding conditions for the terms in $x_1^3x_r$ and $x_1x_r^3$ are

$$\Sigma a=\Sigma a_1a_2a_3=a_1^2a_2a_3+a_1a_2^2a_4+a_1a_3^2a_4+a_2a_3a_4^2$$
$$=a_1^2a_2a_4+a_1a_2a_3^2+a_1a_3a_4^2+a_2^2a_3a_4$$
$$=a_1^2a_3a_4+a_1a_2^2a_3+a_1a_2a_4^2+a_2a_3^2a_4;$$

and those for terms in $x_r^2x_s^2$ are

$$\Sigma a_1a_2=a_1^2a_2a_3+a_1a_2^2a_4+a_1a_3^2a_4+a_2a_3^2a_4+a_1^2a_4^2+a_2^2a_3^2$$
$$=2a_1a_2a_3a_4+a_1^2a_2^2+a_1^2a_3^2+a_2^2a_4^2+a_3^2a_4^2.$$

It will not be necessary to exhibit the rest since the above are sufficient for the determination of a. The first and second indicate that a_1, a_2, a_3 and a_4 are the roots of an equation of the form

$$a^4-Aa^3+Ba^2-Aa+1=0,$$

and, combining them with the others above given, we find

$$\Sigma a\,\Sigma a_1a_2a_3=\Sigma a_1^2a_2a_3+4a_1a_2a_3a_4;$$

or $$A^2=3A+4;\text{ and }A=4,\text{ or }-1.$$

Again:—

$$\Sigma a_1a_2=a_1a_2a_3a_4+\tfrac{1}{2}(\Sigma a\,\Sigma a_1^2a_2^2)$$
$$=\tfrac{1}{2}(\Sigma a_1a_2)^2+\tfrac{1}{2}\Sigma a-\Sigma a_1^2a_2a_3-2a_1a_2a_3a_4;$$

or $$B=\tfrac{1}{2}B^2-\tfrac{5}{2}A-2.$$

Hence the three systems

$$A=4,\ B=6,\text{ or }-4;\ A=-1,\ B=1,$$

give rise respectively to the relations

$$(a-1)^4=0,$$
$$(a-1)^4-10a^2=0,$$

and $$a^4+a^3+a^2+a+1=0.$$

The first renders X symmetric. The second does not enable us to satisfy the third equation of condition. But the roots of the third (which are, in fact, the unreal fifth roots of unity) satisfy them all. Let then i, i^2, i^3 and i^4 denote these roots, and

$$f(i) = x_1 + i\,x_2 + i^2 x_3 + i^3 x_4 + i^4 x_5;$$

then
$$f(i^2) = x_1 + i^2 x_2 + i^4 x_3 + i\,x_4 + i^3 x_5,$$
$$f(i^3) = x_1 + i^3 x_2 + i\,x_3 + i^4 x_4 + i^2 x_5,$$
$$f(i^4) = x_1 + i^4 x_2 + i^3 x_3 + i^2 x_4 + i\,x_5,$$

and
$$\pi_4(x) = f(i)\cdot f(i^2)\cdot f(i^3)\cdot f(i^4).$$

It will be observed that, if from the above expressions we expunge the x's, the four horizontal rows read downwards are identical in value and order with the four vertical columns read from left to right, and that i and i^4 lie in inverse symmetry upon, and i^2 and i^3 around, diagonals. It does not appear to be possible to render $\pi_4(x)$ more nearly symmetrical. A similar arrangement when n is prime, or a modification of it when n is odd, will probably be found available. When all the divisors of n are even, we must adopt an arrangement analagous to that employed for biquadratics. (See Art. 4.)

By actual development, we find

$$f(i)\cdot f(i^4) = \Sigma x^2 + \tau\phi(i) + \tau'\phi(i^2),$$
$$\therefore\ f(i^2)\cdot f(i^3) = \Sigma x^2 + \tau\phi(i^2) + \tau'\phi(i),$$

where
$$\tau = x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 + x_5 x_1,$$
$$\tau' = x_1 x_3 + x_3 x_5 + x_5 x_2 + x_2 x_4 + x_4 x_1,$$

and
$$\phi(i) = i + i^4;$$

$$\therefore\ \pi_4(x) = \{\Sigma x^2 + \tau\phi(i) + \tau'\phi(i^2)\}\cdot\{\Sigma x^2 + \tau\phi(i^2) + \tau'\phi(i)\}\cdot$$

Developing and bearing in mind the known properties of i, and the relation

$$\tau + \tau' = \Sigma x_1 x_2,$$

we find

$$\pi_4(x) = (\Sigma x^2)^2 - \Sigma x^2\,\Sigma x_1 x_2 - (\Sigma x_1 x_2)^2 + 5\tau\tau'$$
$$= (\Sigma x)^4 - 5(\Sigma x)^2\,\Sigma x_1 x_2 + 5(\Sigma x_1 x_2)^2 + 5\tau\tau'.$$

The function $\tau\tau'$ is unsymmetric relatively to x; conse-

quently, when $n=5$, $\pi_{n-1}(x)$ cannot in general be rendered symmetric.

6. To show the bearing of this theory on the solution of equations. Let $x_1, x_2, x_3, \cdots x_n$ represent the roots of an equation of the n th degree,

$$x^n + ax^{n-1} + bx^{n-2} + \cdots + sx + t = 0,$$

and consider the effect of supposing $\pi_{n-1}(x)$ to vanish.

For the quadratic

$$x^2 + ax + b = 0,$$

we have (Art. 2)

$$\pi_1(x) = \sqrt{(x_1 + x_2)^2 - 4x_1 x_2} = \sqrt{a^2 - 4b},$$

and the evanescence of $\pi_1(x)$ gives $b = \frac{1}{4}a^2$,

$$\therefore\ x^2 + ax + \tfrac{1}{4}a^2 = 0, \text{ or } (x + \tfrac{1}{2}a)^2 = 0,$$

which is immediately solvible.

7. For the cubic

$$x^3 + ax^2 + bx + c = 0,$$

we have (Art. 3)

$$\pi_2(x) = (\Sigma x)^2 - 3\Sigma x_1 x_2 = a^2 - 3b,$$

and the evanescence of $\pi_2(x)$ gives $b = \frac{1}{3}a^2$,

$$\therefore\ x^3 + ax^2 + \tfrac{1}{3}a^2 x + c = 0,$$

or

$$(x + \tfrac{1}{3}a)^3 - (\tfrac{1}{27}a^3 - c) = 0,$$

which admits of easy solution.

8. For the biquadratic

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

we have (Art. 4)

$$\pi_3(x) = (\Sigma x)^3 - 4\Sigma x\, \Sigma x_1 x_2 + 8\Sigma x_1 x_2 x_3$$
$$= -a^3 + 4ab - 8c,$$

and the evanescence of $\pi_3(x)$ gives

$$b = \tfrac{1}{4}a^2 + \frac{2c}{a},$$

$$\therefore\ x^4 + ax^3 + \left(\tfrac{1}{4}a^2 + \frac{2c}{a}\right)x^2 + cx + d = 0,$$

or

$$\left(x^2 + \tfrac{1}{2}ax - \frac{c}{a}\right)^2 - \left(\frac{c^2}{a^2} - d\right) = 0,$$

which also admits of being easily resolved.

9. Thus for the lower equations, the evanescence of the symmetric product conducts to a solution. If therefore the result of the elimination of x between

$$x^n + ax^{n-1} + bx^{n-2} + \cdots + sx + t = 0,$$

and

$$y - \psi(x) = 0,$$

where ψ is rational, be represented by

$$y^n + Ay^{n-1} + By^{n-2} + \cdots + Sy + T = 0,$$

and ψ be so constructed that $\pi_{n-1}(y)$ may vanish, y, and consequently x, will become known.

The expression $\pi_{n-1}(p+x)$ does not contain p, and the relation

$$y = \psi(x) = p + x$$

leads to illusory results; but if we assume

$$y = \psi(x) = px + x^2,$$

we can reduce the solution of the general biquadratic to that of a cubic, and the solution of the general cubic to that of a quadratic. For since p only enters the product $\pi_{n-1}(px+x^2)$ to the degree which y attains in $\pi_{n-1}(y)$, that is, to the $(n-1)$th degree, and $\pi_2(px+x^2)$ and $\pi_3(px+x^2)$ are symmetric with respect to x, it follows that the equations

$$\pi_2(px+x^2) = 0, \text{ and } \pi_3(px+x^2) = 0,$$

are respectively of the second and third degree in p, and that their solution gives p, a symmetric function of x, in terms of a, b, &c.

10. For quadratics this method must be modified. It is impossible to construct ψ so that $\pi_1(y)$ may vanish without being led to nugatory results. But in this case the symmetric product yields an immediate solution. For (Art. 2)

$$\pi_1(x) = \pm(x_1 - x_2) = \sqrt{(x_1+x_2)^2 - 4x_1x_2};$$

or

$$x_1 - x_2 = \pm\sqrt{a^2 - 4b}.$$

and

$$x_1 + x_2 = -a,$$

whence the usual solution

$$x = -\tfrac{1}{2}a \pm \tfrac{1}{2}\sqrt{a^2 - 4b}.$$

11. *Cubic Equations.* The elimination of x between

$$x^3+ax^2+bx+c=0=\chi(x),$$

and $$y=\psi(x)=px+x^2,$$

gives $$y^3+Ay^2+By+C=0,$$

where

$$A=-\Sigma y=-(p\Sigma x+\Sigma x^2)=ap-(a^2-2b),$$

$$B=\Sigma y_1 y_2=p^2\Sigma x_1 x_2+p\Sigma x_1^2 x_2+\Sigma x_1^2 x_2^2$$
$$=bp^2-(ab-3c)p-2ac+b^2,$$

and

$$C=-y_1 y_2 y_3=-x_1 x_2 x_3(p^3+p^2\Sigma x+p\Sigma x_1 x_2+x_1 x_2 x_3)$$
$$=c(p^3-ap^2+bp-c)=-c\chi(-p).$$

Again: — The evanescence of the symmetric product gives

$$\pi_2(y)=\pi_2(px+x^2)=0,$$

or $$(a^2-3b)p^2-(2a^3-7ab+9c)p+a^4-4a^2b+6ac+b^2=0;$$

and, if we make

$$a^2-3b=\kappa,\ ab-9c=\lambda,\ \text{and}\ b^2-3ac=\mu,$$

this equation takes the form

$$\kappa p^2-(2a\kappa-\lambda)p+a^2\kappa-a\lambda+\mu=0,$$

whence $$p=\frac{2a\kappa-\lambda\pm\sqrt{\lambda^2-4\kappa\mu}}{2\kappa}.$$

Consequently A, B and C are known; and, since (Art. 7)

$$\left(y+\frac{A}{3}\right)^3=\left(\frac{A}{3}\right)^3-C,$$

$$\therefore\ y=-\frac{A}{3}+\sqrt[3]{\left(\frac{A}{3}\right)^3-C}.$$

Appropriating this root to y_1, the corresponding formulæ for y_2 and y_3 may be obtained by writing ϵ and ϵ^2 before the cubic radical; and, combining the two equations at the head of this article, we are conducted to

$$(y+p^2-ap+b)x-(p-a)y+c=0,$$

and $$x=\frac{(p-a)y-c}{y+p(p-a)+b}.$$

All the roots of the proposed cubic may be obtained by substituting y_1, y_2 and y_3 successively for y in this expression.

12. If, in place of

$$y=\psi(x)=px+x^2,$$

we assume

$$y=\psi(x)=(p+x)^{-1},$$

and eliminate x between this equation and the proposed one, there results an equation in y for which

$$A=-\Sigma y=\frac{3p^2-2ap+b}{\chi(-p)},$$

$$B=\Sigma y_1 y_2=\frac{-3p+a}{\chi(-p)},$$

and

$$C=-y_1 y_2 y_3=\frac{1}{\chi(-p)}.$$

The condition $\pi_2(y)=0$ now gives

$$\kappa p^2-\lambda p+\mu=0,$$

or

$$p=\frac{\lambda\pm\sqrt{\lambda^2-4\kappa\mu}}{2\kappa}.$$

Hence, as before, A, B, C and y are known, and the roots of the proposed cubic are

$$x_1=y_1^{-1}-p,\ x_2=y_2^{-1}-p,\ \text{and}\ x_3=y_3^{-1}-p.$$

13. *Biquadratic Equations.* If we eliminate x between

$$x^4+ax^3+bx^2+cx+d=0,$$

and

$$x^2+px-y=0,$$

there results

$$y^4+Ay^3+By^2+Cy+D=0,$$

where

$$A=-\Sigma y=-(p\Sigma x+\Sigma x^2)=ap-(a^2-2b),$$

$$\begin{aligned}B=\Sigma y_1 y_2&=p^2\Sigma x_1 x_2+p\Sigma x_1^2 x_2+\Sigma x_1^2 x_2^2\\&=bp^2-(ab-3c)p-2ac+b^2+2d,\end{aligned}$$

$$\begin{aligned}C=-\Sigma y_1 y_2 y_3&=-(p^3\Sigma x_1 x_2 x_3+p^2\Sigma x_1^2 x_2 x_3+p\Sigma x_1^2 x_2^2 x_3\\&\qquad+\Sigma x_1^2 x_2^2 x_3^2)\\&=cp^3-(ac-4d)p^2-(3ad-bc)p+2bd-c^2,\end{aligned}$$

and

$$\begin{aligned}D=y_1 y_2 y_3 y_4&=x_1 x_2 x_3 x_4(p^4+p^3\Sigma x+p^2\Sigma x_1 x_2+p\Sigma x_1 x_2 x_3\\&\qquad+x_1 x_2 x_3 x_4)\\&=d(p^4-ap^3+bp^2-cp+d).\end{aligned}$$

The evanescence of the symmetric product gives

$$\pi_3(y)=\pi_3(px+x^2)=0, \text{ or}$$

$$(a^3-4ab+8c)p^3-(3a^4-14a^2b+20ac+8b^2-32d)p^2$$
$$+(3a^5-16a^3b+20a^2c+16ab^2-32ad-16bc)p$$
$$-(a^6-6a^4b+8a^3c+8a^2b^2-8a^2d-16abc+8c^2)=0,$$

a cubic in p.

Again: — The biquadratic in y may be put under the form (Art. 8)

$$\left(y^2+\tfrac{1}{2}\mathrm{A}y-\frac{\mathrm{C}}{\mathrm{A}}\right)^2=\left(\frac{\mathrm{C}}{\mathrm{A}}\right)^2-\mathrm{D},$$

or
$$y=-\frac{\mathrm{A}}{4}\pm\sqrt{\left(\frac{\mathrm{A}}{4}\right)^2+\frac{\mathrm{C}}{\mathrm{A}}}\pm\sqrt{\left(\frac{\mathrm{C}}{\mathrm{A}}\right)^2-\mathrm{D}}.$$

Finally: — Combining the equations

$$x^4+ax^3+bx^2+cx+d=0,$$

and
$$x^2+px-y=0.$$

by the method of the highest common divisor, we find

$$(ay-2py-p^3+ap^2-bp+c)x+y^2+(p^2-ap+b)y+d=0,$$

or
$$x=\frac{y^2+(p^2-ap+b)y+d}{(2p-a)y+p^3-ap^2+bp-c},$$

and, as in cubics, all the values of x may be evolved.

14. Of course ψ admits of an infinite variety of constructions equally available. For instance, if we give to y another form, and assume

$$y=px+x^{-1};$$

then, eliminating x between this equation and the proposed biquadratic, we are conducted to an equation in y for which

$$\mathrm{A}=ap+\frac{c}{d},$$

$$\mathrm{B}=bp^2+\frac{ac-4d}{d}p+\frac{b}{d},$$

$$\mathrm{C}=cp^3-\frac{3ad-bc}{d}p^2+\frac{ab-3c}{d}p+\frac{a}{d},$$

$$\mathrm{D}=dp^4-\frac{2bd-c^2}{d}p^3-\frac{2ac-b^2-2d}{d}p^2+\frac{a^2-2b}{d}p+\frac{1}{d};$$

and the evanescence of the symmetric product now gives

$$(a^3-4ab+8c)p^3-\frac{1}{d}(a^2c+8ad-4bc)p^2$$
$$+\frac{1}{d^2}(4abd-ac^2-8cd)p+\frac{1}{d^3}(8ad^2-4bcd+c^2)=0.$$

Hence p may be determined; A, B, C, D and consequently y become known, as before; and x is given by

$$x=\frac{y^3+apy-dp^3+bp^2-p}{y^3+apy^2+(bp-2)py+cp^2-ap^3},$$

a result obtained, like the corresponding one in the last article, by the aid of the common divisor.

Section II.

The Resolvent Product for Quintics, and a new Cyclical Symbol.

15. For the general equation of the fifth degree,

$$x^5+ax^4+bx^3+cx^2+dx+e=0,$$

the resolvent product is (Art. 5)

$$\pi_4(x)=(\Sigma x)^4-5(\Sigma x)^2\Sigma x_1x_2+5(\Sigma x_1x_2)^2+5\tau\tau'$$
$$=a^4-5a^2b+5b^2+5\tau\tau'.$$

This product is six-valued. For

$$f(i)=i^0x_1+ix_2+i^2x_3+i^3x_4+i^4x_5,$$

and therefore $f(i)$ has $1\cdot2\cdot3\cdot4\cdot5$ or 120 values, which may be formed either by permuting the powers of i or the roots of the quintic in all possible ways. Let us call $f(i)$ the first factor of $\pi_4(x)$. Then each of the values of $f(i)$ may be made a first factor, so that, at first sight, it might seem that $\pi_4(x)$ has 120 values. But by the properties of i,

$$\pi_4(x)=f(i)\cdot f(i^2)\cdot f(i^3)\cdot f(i^4)$$
$$=f(i^2)\cdot f(i^3)^2\cdot f(i^2)^3\cdot f(i^3)^4$$
$$=f(i^3)\cdot f(i^3)^2\cdot f(i^3)^3\cdot f(i^3)^4$$
$$=f(i^4)\cdot f(i^4)^2\cdot f(i^4)^3\cdot f(i^4)^4.$$

Whence it appears that when $f(i^2)$, $f(i^3)$, or $f(i^4)$, becomes the first factor, no new values of $\pi_4(x)$ are introduced, but we are remitted to our former value. Thus the number is reduced one-fourth. Again: — Let

$$f'(i)=ix_1+i^2x_2+i^3x_3+i^4x_4+i^0x_5=if(i),$$

then $f'(i^2)=i^2f(i^2)$, $f'(i^3)=i^3f(i^3)$, and $f'(i^4)=i^4f(i^4)$;

$$\therefore f'(i)\cdot f'(i^2)\cdot f'(i^3)\cdot f'(i^4)=f(i)\cdot f(i^2)\cdot f(i^3)\cdot f(i^4).$$

That is, when $f'(i)$, or $if(i)$, is made the first factor, no new values of $\pi_4(x)$ are introduced, but we are remitted to our former value. A similar argument holds for $i^2f(i)$, $i^3f(i)$, and $i^4f(i)$. So that the number of values is still further reduced one-fifth, and $\pi_4(x)$ has only $120\div4\cdot5$ or 6 values.*

16. The same result may be arrived at by considering the form of the product, without reference to its factors. For

$$\tau=x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1$$

is a circular function, each term being derived from the preceding by advancing the roots a step in a certain cycle

$$\cdots x_1x_2x_3x_4x_5x_1x_2x_3x_4x_5\cdots$$

And, since the coefficients of the terms in the expression for τ are equal, we may, in forming its values, regard one of the roots, say x_1, as fixed, while the others are permuted *inter se*. Thus we shall have $1\cdot2\cdot3\cdot4$ cycles, giving rise to 24 corresponding expressions for τ; but since

$$\begin{aligned}\tau&=x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1\\&=x_1x_5+x_2x_1+x_3x_2+x_4x_3+x_5x_4,\end{aligned}$$

these 24 expressions may be grouped in pairs, the members of each pair being equal. Hence τ has only 12 values. Again: — Since

$$\begin{aligned}\tau'&=x_1x_3+x_3x_5+x_5x_2+x_2x_4+x_4x_1\\&=x_1x_3+x_2x_4+x_3x_5+x_4x_1+x_5x_2,\end{aligned}$$

the several values of τ' may be referred to the same 12 cycles that arise in the formation of the values of τ. Con-

* For this proof I am indebted to my friend Mr. Cockle; and I may add that, having from time to time sent him portions of my researches, he has been so good as to give me the benefit of his more extensive reading on the subject, to point out coincidences, to suggest modifications, and in various other ways to help me to improve my methods and abridge my calculations.

sequently, since τ and τ' are complementary to each other (for their sum $=\Sigma x_1 x_2 = b$, a one-valued function), it follows that $\tau\tau'$, and therefore

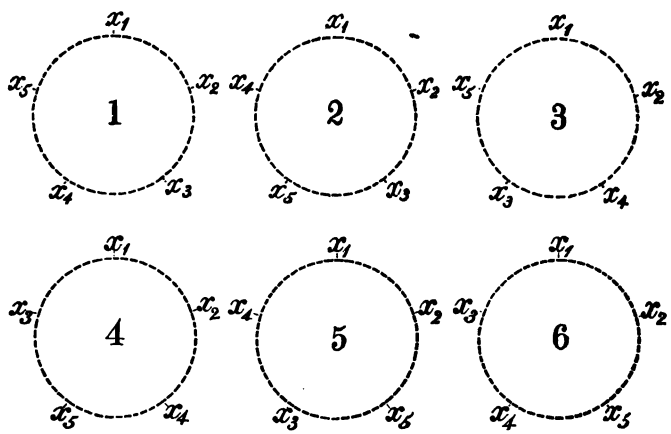
$$a^4 - 5a^2 b + 5b^2 + 5\tau\tau',$$

or $\pi_4(x)$, is, as we have otherwise proved in the last article, a six-valued function.

17. The six values of $\tau\tau'$ may be exhibited in two ways. First,

$$(x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1)\cdot(x_1x_3+x_2x_4+x_3x_5+x_4x_1+x_5x_2),$$
$$(x_1x_2+x_2x_3+x_3x_5+x_5x_4+x_4x_1)\cdot(x_1x_3+x_2x_5+x_3x_4+x_5x_1+x_4x_2),$$
$$(x_1x_2+x_2x_4+x_4x_3+x_3x_5+x_5x_1)\cdot(x_1x_3+x_2x_5+x_4x_1+x_3x_2+x_5x_4),$$
$$(x_1x_2+x_2x_4+x_4x_5+x_5x_3+x_3x_1)\cdot(x_1x_4+x_2x_5+x_4x_3+x_5x_1+x_3x_2),$$
$$(x_1x_2+x_2x_5+x_5x_3+x_3x_4+x_4x_1)\cdot(x_1x_3+x_2x_4+x_5x_1+x_3x_2+x_4x_5),$$
$$(x_1x_2+x_2x_5+x_5x_4+x_4x_3+x_3x_1)\cdot(x_1x_4+x_2x_3+x_5x_1+x_4x_2+x_3x_5),$$

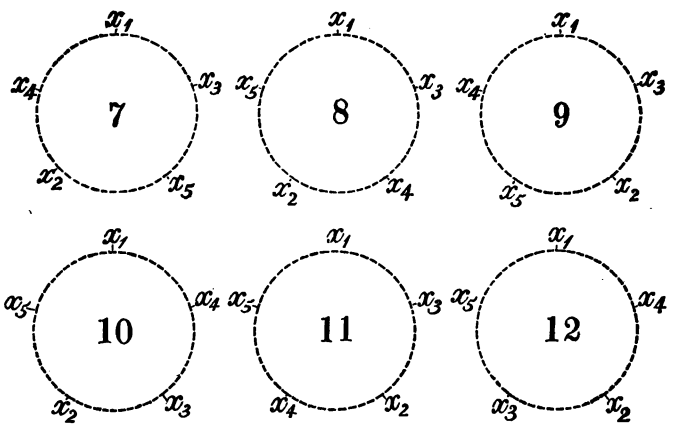
corresponding respectively to the cycles



or, secondly,

$$(x_1x_2+x_3x_4+x_5x_1+x_2x_3+x_4x_5)\cdot(x_1x_3+x_3x_5+x_5x_2+x_2x_4+x_4x_1),$$
$$(x_1x_2+x_3x_5+x_4x_1+x_2x_3+x_5x_4)\cdot(x_1x_3+x_3x_4+x_4x_2+x_2x_5+x_5x_1),$$
$$(x_1x_2+x_3x_5+x_2x_4+x_5x_1+x_4x_3)\cdot(x_1x_3+x_3x_2+x_2x_5+x_5x_4+x_4x_1),$$
$$(x_1x_2+x_4x_5+x_3x_1+x_2x_4+x_5x_3)\cdot(x_1x_4+x_4x_3+x_3x_2+x_2x_5+x_5x_1),$$
$$(x_1x_2+x_3x_4+x_2x_5+x_4x_1+x_5x_3)\cdot(x_1x_3+x_3x_2+x_2x_4+x_4x_5+x_5x_1),$$
$$(x_1x_2+x_4x_3+x_2x_5+x_3x_1+x_5x_4)\cdot(x_1x_4+x_4x_2+x_2x_3+x_3x_5+x_5x_1),$$

corresponding respectively to the cycles



18. *New Cyclical Symbol.* Let $\chi(0)$ be a function of x, and let $\chi(q)$ be derived from $\chi(0)$, by advancing each of the roots contained in it q steps in the cycle r. And let

$$\Sigma'_r \chi(0) = \chi(0) + \chi(1) + \chi(2) + \chi(3) + \chi(4).$$

Then

$$\Sigma'_r x_1^n = \Sigma x^n, \quad \Sigma'_1 x_1^n (x_2^n + x_3^n) = \Sigma x_1^n x_2^n, \text{ \&c.,}$$

and

$$\begin{aligned}
&\Sigma'_1 x_1 x_2 = \Sigma'_7 x_1 x_2, && \Sigma'_1 x_1 x_3 = \Sigma'_7 x_1 x_3,\\
&\Sigma'_2 x_1 x_2 = \Sigma'_8 x_1 x_2, && \Sigma'_2 x_1 x_3 = \Sigma'_8 x_1 x_3,\\
&\Sigma'_3 x_1 x_2 = \Sigma'_9 x_1 x_2, && \Sigma'_3 x_1 x_3 = \Sigma'_9 x_1 x_3,\\
&\Sigma'_4 x_1 x_2 = \Sigma'_{10} x_1 x_2, && \Sigma'_4 x_1 x_4 = \Sigma'_{10} x_1 x_4,\\
&\Sigma'_5 x_1 x_2 = \Sigma'_{11} x_1 x_2, && \Sigma'_5 x_1 x_3 = \Sigma'_{11} x_1 x_3,\\
&\Sigma'_6 x_1 x_2 = \Sigma'_{12} x_1 x_2, && \Sigma'_6 x_1 x_4 = \Sigma'_{12} x_1 x_4.
\end{aligned}$$

The six values of $\tau\tau'$ may now be briefly expressed as follows:

$$\begin{aligned}
&\tau_1 \tau'_1 = \Sigma'_1 x_1 x_2 \Sigma'_1 x_1 x_3 = \Sigma'_7 x_1 x_2 \Sigma'_7 x_1 x_3 = \tau_7 \tau'_7,\\
&\tau_2 \tau'_2 = \Sigma'_2 x_1 x_2 \Sigma'_2 x_1 x_3 = \Sigma'_8 x_1 x_2 \Sigma'_8 x_1 x_3 = \tau_8 \tau'_8,\\
&\tau_3 \tau'_3 = \Sigma'_3 x_1 x_2 \Sigma'_3 x_1 x_3 = \Sigma'_9 x_1 x_2 \Sigma'_9 x_1 x_3 = \tau_9 \tau'_9,\\
&\tau_4 \tau'_4 = \Sigma'_4 x_1 x_2 \Sigma'_4 x_1 x_4 = \Sigma'_{10} x_1 x_2 \Sigma'_{10} x_1 x_4 = \tau_{10} \tau'_{10},\\
&\tau_5 \tau'_5 = \Sigma'_5 x_1 x_2 \Sigma'_5 x_1 x_3 = \Sigma'_{11} x_1 x_2 \Sigma'_{11} x_1 x_3 = \tau_{11} \tau'_{11},\\
&\tau_6 \tau'_6 = \Sigma'_6 x_1 x_2 \Sigma'_6 x_1 x_4 = \Sigma'_{12} x_1 x_2 \Sigma'_{12} x_1 x_4 = \tau_{12} \tau'_{12}.
\end{aligned}$$

Σ'_r must not be confounded with Mr. Cockle's Σ' of

epimetrics, being more general in its conception, and possessing peculiar working properties. It is a symbol of cyclical operation.

19. Since each root recurs at every fifth step in any given cycle, we have

$$\chi'5 = \chi'0 \text{ , and}$$

$$\Sigma_1\chi'1 = \chi'1 + \chi'2 + \chi'3 + \chi'4) + \chi(5) = \Sigma'_1\chi(0);$$

whence, by generalization and induction,

$$\Sigma_1\chi'q = \Sigma_1\chi'0 .$$

That is the circular function $\Sigma_1\chi'0$ is not affected in value by the simultaneous advancing or receding of the roots which it contains any number of steps (q) in the cycle. Consequently

$$\Sigma_1\chi'\Sigma_1\chi'$$
$$-\Sigma_1\chi'[\chi'1 - \chi'2 - \chi'3 + \chi'4)]$$
$$-\Sigma_1\chi'[\chi'1 - \chi'2 - \chi'3 - \chi'4)].$$

The commutative property will be found of great practical use in dealing with circular functions.

20. For example. Taking the first cycle as a type of the rest, and omitting, for the sake of simplicity, the unit

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

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Results, corresponding to a certain extent with those above given, would have been obtained if, instead of confining ourselves to five, we had dealt with any number of symbols.

21. We now proceed to show that, when the resolvent product $\pi_4(x)$ vanishes, the general equation of the fifth degree admits of finite algebraic solution. In establishing the analagous proposition for the lower equations (Art. 6, 7, 8), we supposed the product $\pi_{n-1}(x)$ to vanish, without distinguishing its factors. But by assuming the evanescence of a factor we should of course be conducted to substantially the same results, and in dealing with the higher equations it will be found that the latter method possesses peculiar advantages over the former. In order to abridge the calculations, we shall suppose the given equation to be deprived of its second term; *i.e.* that

$$-a=x_1+x_2+x_3+x_4+x_5=0,$$

a supposition that will not affect any of the values of $f(i^n)$. Next, if we make

$$\begin{aligned} f(i) &= x_1+i\,x_2+i^2x_3+i^3x_4+i^4x_5=0,\\ f(i^2) &= x_1+i^2x_2+i^4x_3+i\,x_4+i^3x_5=5\beta_1,\\ f(i^3) &= x_1+i^3x_2+i\,x_3+i^4x_4+i^2x_5=5\beta_2,\\ \text{and}\quad f(i^4) &= x_1+i^4x_2+i^3x_3+i^2x_4+i\,x_5=5\beta_3, \end{aligned}$$

we shall have

$$\begin{aligned} x_1&=\beta_1+\beta_2+\beta_3, & x_4&=i^4\beta_1+i\,\beta_2+i^3\beta_3,\\ x_2&=i^3\beta_1+i^2\beta_2+i\beta_3, & x_5&=i^2\beta_1+i^3\beta_2+i^4\beta_3,\\ x_3&=i\,\beta_1+i^4\beta_2+i^2\beta_3, & \text{and } b&=\Sigma x_1x_2=-5\beta_1\beta_2. \end{aligned}$$

Whence we find

$$\Sigma x^3=15\beta_1\beta_3^2+15\beta_2^2\beta_3,$$

$$\Sigma x^4=20\beta_1^3\beta_3+30\beta_1^2\beta_2^2+20\beta_2\beta_3^3,$$

$$\text{and}\quad \Sigma x^5=5\beta_1^5+150\beta_1^2\beta_2\beta_3^2+100\beta_1\beta_2^3\beta_3+5\beta_2^5+5\beta_3^5.$$

But by the method of the limiting equation, or otherwise, we obtain

$$\Sigma x^3=-3c,\ \Sigma x^4=2b^2-4d,\ \text{and}\ \Sigma x^5=5(bc-e);$$

and by comparison with the above,

$$-c=5\beta_1\beta_2^2+5\beta_3^2\beta_4,$$
$$-d=5\beta_1^3\beta_4+5\beta_2\beta_3^3-\tfrac{1}{3}b^2,$$
and $$-e=\beta_1^5+\beta_2^5+\beta_3^5-b(\beta_1\beta_2^2-\beta_3^2\beta_4).$$

Or,* if we put

$$-b=5\mathrm{P},\ -c=5\mathrm{Q},\ -d+\tfrac{1}{3}b^2=5\mathrm{S},$$
$$e=\mathrm{E},\ \beta_2^2\beta_3=u,\ \text{and}\ \beta_3^5=\mathrm{P}v,$$

we shall have

$$u^2+vu-\mathrm{Q}v=0 \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$u^3+\mathrm{P}^3u-\mathrm{PS}v=0 \quad . \quad . \quad . \quad . \quad (2),$$
$$u^5-10\mathrm{P}^3v^2u+\mathrm{P}^3v^3+\mathrm{P}^2(5\mathrm{PQ}+\mathrm{E})v^2+\mathrm{P}^6v=0 \quad . \quad (3).$$

The elimination of v between (1) and (2) gives

$$u^3-\mathrm{Q}u^2+\mathrm{P}(\mathrm{P}^2+\mathrm{S})u-\mathrm{P}^3\mathrm{Q}=0 \quad . \quad . \quad (4),$$

and between (1) and (3),

$$u^6-3\mathrm{Q}u^5-(11\mathrm{P}^3-3\mathrm{Q}^2)u^4+(\mathrm{P}^2\mathrm{E}+15\mathrm{P}^3\mathrm{Q}-\mathrm{Q}^3)u^3$$
$$-\mathrm{P}^2(\mathrm{QE}+\mathrm{P}^4+5\mathrm{PQ}^2)u^2+2\mathrm{P}^6\mathrm{Q}u-\mathrm{P}^6\mathrm{Q}^2=0 \quad . \quad (5).$$

Proceeding with (4) and (5) by division, we obtain

$$\mathfrak{a}u^2+\mathfrak{b}u+\mathfrak{c}=0 \quad . \quad . \quad . \quad . \quad (6),$$

where

$$\mathfrak{a}=11\mathrm{P}^4+13\mathrm{P}^2\mathrm{S}-2\mathrm{PQ}^2+\mathrm{S}^2,$$
$$\mathfrak{b}=-\mathrm{P}(\mathrm{P}^2+\mathrm{S})\mathrm{E}-\mathrm{Q}(16\mathrm{P}^4+8\mathrm{P}^2\mathrm{S}-\mathrm{PQ}^2+\mathrm{S}^2),$$
and $$\mathfrak{c}=\mathrm{P}^2\mathrm{Q}(\mathrm{PE}+5\mathrm{P}^2\mathrm{Q}+\mathrm{QS}).$$

We might now carry on the process of division until we arrived at a linear equation in u, but we shall be conducted to a more simple and elegant result by proceeding thus: — Arrange the terms in (4) and (5) according to ascending powers of u, divide the latter equation by u^2, and the result by the former, until there arises a biquadratic in u. Arrange the terms of this equation according to descending powers of u, and divide it by (4); there will result a quadratic in u. In effect, we shall be conducted to

* Mr. Cockle's notation is here adopted, so far as my own objects will allow, in order to facilitate the comparison of our results. See a series of papers by that gentleman "On Equations of the Fifth Degree," published in the appendices of the *Lady's and Gentleman's Diary* for 1848, 1851, 1856, 1857 and 1858.

$$Au^3+Bu+C=0 \quad . \quad . \quad . \quad . \quad (7),$$

where

$$A=11P^3+2PS-Q^2,$$
$$B=-P^2E-Q(16P^3+3PS-Q^2),$$

and $C=P^2QE+P^2(6PQ^2+S)$.

By (6, 7),

$$(\mathfrak{a}-AP)u^3+(\mathfrak{b}-BP)u+(\mathfrak{c}-CP)=0,$$

or

$$A'u^3+B'u+C'=0 \quad . \quad . \quad . \quad . \quad (8),$$

where

$$A'=11P^2S-PQ^2+S^2,$$
$$B'=-PSE-QS(5P^2+S),$$

and $C'=-P^2(P^2Q^2+PS^2-Q^2S)$.

Combining (7) and (8), we find

$$(A'B-AB')u+(A'C-AC')=0,$$

or

$$P(mE+n)u-P(pE+q)=0,$$

$$\therefore\ u=\frac{pE+q}{mE+n},$$

where

$$m=P^2Q^2+PS^2-Q^2S,$$
$$n=-Q(121P^4S-16P^3Q^2+28P^2S^2-9PQ^2S+Q^4+S^3),$$
$$p=-PQ(11P^2S-PQ^2+S^2), \text{ and}$$
$$q=-P(11P^5Q^2+11P^4S^2+57P^3Q^2S-7P^2Q^4+13P^2S^3$$
$$+2PQ^2S^2+Q^4S+S^4).$$

Moreover, multiplying (1) into PS, and (2) into Q, subtracting the former from the latter result, and dividing the difference by u, we obtain

$$Qu^3-PSu+P(P^2Q-Sv)=0 \quad . \quad . \quad (9).$$

Next, multiplying (1) into Q, and subtracting (9) from the result, we have

$$(PS-Q^2+Qu)v-P(P^2Q-Su)=0,$$

$$\therefore\ v=\frac{P(P^2Q-Su)}{PS-Q^2+Qu}=\frac{P(p'E+q')}{m'E+n'},$$

where

$$m'=-S(10P^3Q^2-P^2S^2+3PQ^2S-Q^4),$$
$$n'=-Q(11P^6Q^2+132P^5S^2-80P^4Q^2S+9P^3Q^4+41P^3S^3$$
$$-35P^2Q^2S^2+11PQ^4S+2PS^4-Q^6-Q^2S^3),$$

$$p' = PQ(P^3Q^3 + 12P^2S^3 - 2PQ^2S + S^3), \text{ and}$$
$$q' = -P(110P^5Q^3S - 16P^4Q^4 - 11P^4S^3 - 29P^3Q^3S^2 - 2P^2Q^4S - 13P^2S^4 + PQ^6 - PQ^2S^3 - Q^4S^2 - S^5).$$

The quantities u and v being now known, β_1, β_2 and β_3 are given by

$$\beta_1 = \sqrt[5]{\frac{P^4}{v}}, \quad \beta_2 = \sqrt[5]{Pv}, \text{ and } \beta_3 = \frac{u}{\sqrt[5]{P^2v^3}},$$

and the roots of the quintic

$$x^5 - 5Px^3 - 5Qx^2 - 5Rx + E = 0 \quad . \quad . \quad (10),$$

where

$$R = -\tfrac{1}{5}c = S - P^2,$$

are the values of the expression

$$(1)^{\frac{1}{5}}\sqrt[5]{\frac{P^4}{v}} + (1)^{\frac{2}{5}}\sqrt[5]{Pv} + (1)^{\frac{3}{5}}\frac{u}{\sqrt[5]{P^2v^3}}.$$

22. Substituting for u its value in (8), we obtain

$$A'(pE+q)^2 + B'(pE+q)(mE+n) + C'(mE+n)^2 = 0,$$

a relation which admits of simplification. In fact, if we restore the values of A′, B′, C′, m, n, p and q in terms of P, Q and S, that relation takes the form

$$A'm(PQSE^3 + \kappa E^2 + \lambda E + \mu) = 0,$$

where κ, λ and μ are rational and integral functions of P, Q and S.

$$\therefore\ PQSE^3 + \kappa E^2 + \lambda E + \mu = 0;$$

or, in effect,

$$PQSE^5 + \{P^4Q^2 + P^3S^2 + 4P^2Q^2S + PQ^4 + PS^3 + Q^2S^2\}E^2 + \{121P^6QS + 10P^5Q^3 + 175P^4QS^2 - 88P^3Q^2S + 12P^2Q^5 + 27P^2QS^3 - 7PQ^3S^2 + 2Q^5S + 2QS^4\}E + 121P^6Q^2 + 121P^3S^2 - 561P^7Q^2S + 102P^6Q^4 + 286P^6S^3 + 227P^5Q^2S^2 - 137P^4Q^4S + 191P^4S^4 + 17P^3Q^6 - 30P^3Q^3S^3 + 8P^2Q^4S^2 + 26P^2S^5 - 7PQ^2S^4 - 7PQ^6S + Q^8 + 2Q^4S^3 + S^6 = 0 \quad . \quad (11),$$

a relation due to Mr. Cockle, who first announced it in the Appendix to the *Lady's and Gentleman's Diary* for last year, p. 82. Of quintics, whose roots have, as yet, been exhibited,* this is the most comprehensive form. It

* Mr. Cockle, in the Paper above referred to, does not *exhibit* the roots

includes, as particular cases, the quadrinomials of De Moivre and Euler, and the ordinary binomial.

23. Thus comparing De Moivre's equation

$$x^5 - 5ax^3 + 5a^2x - 2\gamma = 0 \quad . \quad . \quad . \quad (12)$$

with (10), we see that

$$P = a,\ Q = 0,\ R = -a^2, \text{ and } E = -2\gamma,$$
$$\therefore P^2 = -R, \text{ and } S = P^2 + R = 0.$$

Now when Q and S vanish, the condition (11) is satisfied, and

$$\beta_1\beta_2 = P,\ \beta_1\beta_3^2 + \beta_2^2\beta_3 = 0,\ \beta_1^3\beta_3 + \beta_2\beta_3^3 = 0, \text{ and}$$
$$\beta_1^5 + \beta_2^5 + \beta_3^5 + \tfrac{5}{2}P(\beta_1\beta_3^2 - \beta_2^2\beta_3) = -E.*$$

The second and third of these equations are satisfied by $\beta_3 = 0$, and the fourth becomes

$$\beta_1^5 + \beta_2^5 = -E,$$

which, combined with the first, gives

$$\beta_1^5 = -\tfrac{1}{2}E + \sqrt{\tfrac{1}{4}E^2 - P^5} = \gamma + \sqrt{\gamma^2 - a^5},$$
$$\beta_2^5 = -\tfrac{1}{2}E - \sqrt{\tfrac{1}{4}E^2 - P^5} = \gamma - \sqrt{\gamma^2 - a^5};$$

and the roots of (12) are the five values of the expression

$$(1)^{\frac{1}{5}}\sqrt[5]{\gamma + \sqrt{\gamma^2 - a^5}} + (1)^{\frac{2}{5}}\sqrt[5]{\gamma - \sqrt{\gamma^2 - a^5}}, \text{ or}$$
$$(1)^{\frac{1}{5}}\sqrt[5]{\gamma + \sqrt{\gamma^2 - a^5}} + \frac{a(1)^{\frac{2}{5}}}{\sqrt[5]{\gamma + \sqrt{\gamma^2 - a^5}}}.$$

24. Next, when P vanishes, S = R, and (11) becomes

$$(QRE + Q^4 + R^3)^2 = 0, \text{ or}$$
$$QRE + Q^4 + R^3 = 0,$$

Euler's criterion. In this case (Art. 21)

$$\beta_1\beta_2 = 0,\ \beta_1\beta_3^2 + \beta_2^2\beta_3 = Q,\ \beta_1^2\beta_3 + \beta_2\beta_3^2 = R,$$
and $$\beta_1^5 + \beta_2^5 + \beta_3^5 = -E.$$

of the quintic, but they are implicitly given. The value of u, which may be deduced from Mr. Cockle's equations, will be found to differ *in form* from that given in the text (Art. 21), a circumstance arising from the difference in our methods of eliminating v and of combining the eliminants.

* It is to be noticed that these equations are employed in order to avoid the vanishing fractions which would arise in attempting to deduce the roots immediately from the formulæ given in Art. 21.

From the first equation we have β_1 or $\beta_2=0$, and the second and third give

$$\beta_2^5 \text{ or } \beta_3^5=\frac{Q^3}{R}, \text{ and } \beta_3^5 \text{ or } \beta_1^5=\frac{R^3}{Q},$$

which satisfy the fourth. For

$$\frac{Q^3}{R}+\frac{R^3}{Q}=-E,$$

or
$$QRE+Q^4+R^3=0.$$

Consequently, when this relation holds, the roots of the quintic

$$x^5-5Qx^3-5Rx+E=0,$$

are all included in the formula

$$(1)^{\frac{3}{5}}\sqrt[5]{\frac{Q^3}{R}}+(1)^{\frac{1}{5}}\sqrt[5]{\frac{R^3}{Q}}\ .$$

25. Again: — When P, Q and R vanish, (11) is satisfied, and

$$x=(1)^{\frac{1}{5}}\sqrt[5]{-E},$$

the solution of the ordinary binomial form.

26. To return to the resolvent product. Denote $\pi_4(x)$ by θ, and put

$$\Sigma' x_1^3(x_2x_5+x_3x_4)=U;$$

then (Art. 20)

$$\tau\tau'=\Sigma x_1x_2x_3x_4+\Sigma x_1^3x_2x_3-U=ac-d-U,$$

and
$$\theta+5U=a^4-5a^3b+5ac+5b^3-15d.$$

Hence, for the equation

$$x^5-5Mx^4-5Px^3-5Qx^2-5Rx+E=0,$$

we have

$$\tau\tau'=5^2MQ+3\cdot5R-U, \text{ and}$$

$$\theta+5U=5^4M^4+5^4M^2P+5^3MQ+5^3P^2+3\cdot5^2R.$$

Now it is known that the general equation of the fifth degree may be put under any one of the six following forms, viz.,

$$x^5-5Px^3-5Qx^2-5Rx+E=0 \quad . \ . \ . \quad (a)$$

$$x^5-5Qx^3-5Rx+E=0 \quad . \ . \ . \quad (b)$$

$$x^5 - 5Rx + E = 0 \quad . \quad . \quad . \quad (c)$$
$$x^5 - 5Qx^2 + E = 0 \quad . \quad . \quad . \quad (d)$$
$$x^5 - 5Px^3 + E = 0 \quad . \quad . \quad . \quad (e)$$
$$x^5 - 5Mx^4 + E = 0 \quad . \quad . \quad . \quad (f)$$

(See Mr. Jerrard's "Mathematical Researches;" Sir W. R. Hamilton's "Inquiry into the Validity of Mr. Jerrard's Method," published in the Sixth Report of the British Association for the Advancement of Science; and M. Serret's "Cours d'Algèbre Supérieure," Note V.)

For (a), (b) and (c), we have

$$\tau\tau' = -\tau^3 = 3 \cdot 5R - U, \text{ or } U = \tau^3 + 3 \cdot 5R \ . \ (a);$$

and the corresponding relation for (d), (e) and (f) is

$$\tau\tau' = -\tau^3 = -U, \text{ or } U = \tau^3 \quad . \quad . \quad (a').$$

Again: — For (a), we have

$$\theta + 5U = 5^3P^2 + 3 \cdot 5^2R \quad . \quad . \quad . \quad (\beta);$$

the corresponding relation for (b) and (c) is

$$\theta + 5U = 3 \cdot 5^2R \quad . \quad . \quad . \quad . \quad (\beta');$$

for (d), the relation is

$$\theta + 5U = 0 \quad . \quad . \quad . \quad . \quad (\beta'');$$

and for (e) and (f),

$$\theta + 5U = 5^3P^2 \quad , \quad . \quad . \quad . \quad (\beta''')$$

and

$$\theta + 5U = 5^4M^4 \quad . \quad . \quad . \quad . \quad (\beta^{iv})$$

respectively.

It hence appears that (d) will be the most convenient form with which to deal in calculating the coefficients of the equation in θ.

27. We know that, for (a),

$$\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6 = k(PQSE^3 + \kappa E^2 + \lambda E + \mu),$$

where k is a numerical constant, and κ, λ and μ have the same signification as in Art. 22. To determine k, let us take the particular equation

$$x^5 - 5^3x^2 = 0;$$

then, making $x_1 = 0$, $x_2 = 0$, $x_3 = 5$, $x_4 = 5\epsilon$, and $x_5 = 5\epsilon^2$, (ϵ denoting, as in former articles, an unreal cube root of unity) we find

$$U_1=\Sigma'_1 x_1^3(x_2x_5+x_3x_4)=x_3x_4^3x_5=5^4\epsilon;$$

$\therefore$ by (β'), $\qquad \theta_1=-5U_1=-5^5\epsilon;$

and similarly

$$\theta_2=\theta_4=-5^5\epsilon^3,\ \theta_3=\theta_5=-5^5, \text{ and } \theta_6=-5^5\epsilon;$$

$$\therefore\ \theta_1\theta_2\theta_3\theta_4\theta_5\theta_6=5^{30}.$$

But in the particular case now under notice

$E=0$, and $\mu=Q^3=5^{15}$; $\therefore\ \theta_1\theta_2\theta_3\theta_4\theta_5\theta_6=k\mu=5^{15}k$;

consequently $\qquad 5^{15}k=5^{30}$, or $k=5^{15}$.

Generally, therefore,

$$\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6=5^{15}(PQSE^3+\kappa E^2+\lambda E+\mu),$$

which gives for (d), by restoring the values of κ, λ and μ and making P and S vanish,

$$\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6=5^{15}Q^3.$$

Section III.

Direct Calculation of the Equation in θ.

28. The six values of θ may be regarded as the roots of the equation

$$\theta^6+D_1\theta^5+D_2\theta^4+D_3\theta^3+D_4\theta^2+D_5\theta+D_6=0 \quad . \quad (1).$$

Suppose now that, by Mr. Jerrard's method, or otherwise, the general equation of the fifth degree is reduced to the form

$$x^5-5Qx^2+E=0 \quad . \quad . \quad . \quad . \quad . \quad (2),$$

then (Art. 26)

$$\theta=-5\tau^3=-5U, \text{ and}$$

$$\therefore\ D_1=-\Sigma\theta=5\Sigma\tau^3=5\Sigma U.$$

We might, therefore, calculate this coefficient by writing out the six values of $-\theta$, $5\tau^3$, or $5U$, and taking their sum; but the labour may be materially abridged by the following method. (See Section II., Art. 18, *et seq.*)

$$\tau^3=(\Sigma' x_1x_2)^2=\Sigma' x_1x_2\tau$$

$$=\Sigma'(x_1^3x_2^3+x_1x_2^3x_3+x_1x_2x_3x_4+x_1x_2x_4x_5+x_1^3x_2x_5)$$

$$=\Sigma'(x_1^3x_2^3+2x_1^3x_2x_5)+2\Sigma x_1x_2x_3x_4;$$

or, since $\Sigma x_1x_2x_3x_4=0$,

$$\tau^3=\Sigma'(x_1^3x_2^3+2x_1^3x_2x_5).$$

Now the symbol Σ' written without a suffix being understood to belong to the first cycle (which is taken as a type of the rest), we may obtain $\Sigma\tau^3$ without developing the function on the right hand side of this equation, or finding the values of τ^3 corresponding to the other cycles. The following theorem enables us to pass directly from τ_r^{2n} or U_r^n to $\Sigma\tau^{2n}$ or ΣU^n.

29. Let

$$\tau_r^{2n} \text{ or } U_r^n = \Sigma'_r(X_a + X_b + \cdots + X_s + X_t),$$

X being defined by

$$X_m = \mathfrak{a}' x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta x_5^\epsilon + \mathfrak{a}'' x_1^\beta x_2^\alpha x_3^\gamma x_4^\delta x_5^\epsilon + \mathfrak{a}''' \text{ \&c.},$$

where a, β, γ, δ and ϵ are positive integers or (some of them) zero; and m is the number of values of $x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta x_5^\epsilon$. Then if we assume

$$\Sigma X_m = (\mathfrak{a}' + \mathfrak{a}'' + \mathfrak{a}''' + \text{\&c.})\Sigma x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta x_5^\epsilon,$$

we shall have

$$\Sigma\tau^{2n} \text{ or } \Sigma U^n = \frac{30}{a}\Sigma X_a + \frac{30}{b}\Sigma X_b + \cdots + \frac{30}{s}\Sigma X_s + \frac{30}{t}\Sigma X_t.$$

For, let $\dot\Sigma$ represent the sum of the 24 expressions for τ^{2n} or U^n, formed by applying all the cycles* (Art. 17) to any one of the values of τ^{2n} or U^n. Then, since $\Sigma'_r X_m$ consists of 5, and consequently $\dot\Sigma\Sigma'_r X_m$ of $5\cdot 24$, or 120, expressions of the form X_m, and since $m = 120$, or a sub-multiple of 120, it follows that

$$\dot\Sigma\tau^{2n} \text{ or } \dot\Sigma U^n = \frac{120}{a}\Sigma X_a + \frac{120}{b}\Sigma X_b + \cdots + \frac{120}{s}\Sigma X_s + \frac{120}{t}\Sigma X_t.$$

But since τ^{2n} or U^n is six-valued, we have $\dot\Sigma\tau^{2n} = 4\Sigma\tau^{2n}$, and $\dot\Sigma U^n = 4\Sigma U^n$. Hence the theorem.

30. Applying this: — Since (Art. 28)

$$\tau^3 = \Sigma'(x_1^3 x_2^3 + 2x_1^3 x_2 x_5),$$

$$\therefore\ \Sigma\tau^3 = \frac{30}{10}\Sigma x_1^3 x_2^3 + \frac{2\cdot 30}{30}\Sigma x_1^3 x_2 x_5$$

$$= 3\Sigma x_1^3 x_2^3 + 2\Sigma x_1^3 x_2 x_3 = 0.$$

* In Art. 17 only 12 cycles are exhibited, but the remaining 12 are given by reading the symbols in a reverse order.

Or, working with U, in place of τ^3,

$$U=\Sigma' x_1^3(x_2x_5+x_3x_4);$$

$$\therefore\ \Sigma U=\frac{30}{30}(1+1)\Sigma x_1^3x_2x_3=2\Sigma x_1^3x_2x_3=0,$$

which confirms the value of $\Sigma\tau^3$. Consequently,

$$D_1=5\Sigma\tau^3=5\Sigma U=0.$$

31. Again: — By the commutative property established in Art. 19,

$$\tau^4=\Sigma'(x_1^3x_2^3+2x_1^3x_2x_5)\tau^3;$$

and developing τ^3 according to descending powers of x, we have

$$\tau^3=x_1^3x_2^3+2x_1^3x_2x_5+x_1^3x_5^3+2x_1x_2^3x_3+2x_1x_4x_5^3+x_2^3x_3^3 \\ +2x_2x_3^3x_4+x_3^3x_4^3+2x_3x_4^3x_5+x_4^3x_5^3;$$

whence, multiplying by $x_1^3x_2^3+2x_1^3x_2x_5$, reducing by means of such relations as (Art. 19)

$$\Sigma'x_1^3x_2^4x_3=\Sigma''x_1^4x_2x_5^3,\ \Sigma'x_1^3x_2x_4x_5^3=\Sigma''x_1^3x_2^3x_3x_5,\ \&c.$$

and collecting similar terms, we find

$$\tau^4=\Sigma''\{x_1^4x_2^4+4(x_1^4x_2^3x_5+x_1^4x_3x_5^3)+6x_1^4x_2^2x_5^2+8x_1^3x_2^3x_3x_5 \\ +4(x_1^3x_2^3x_3x_5^2+x_1^3x_2^2x_4x_5^3)+2x_1^3x_2^2x_3^2x_4^2+8x_1^3x_2^2x_3^2x_4x_5 \\ +4x_1^3x_2^2x_3x_4^2x_5\};$$

or, since

$$\Sigma''x_1^3x_2^2x_3^2x_4^2=\Sigma x_1^3x_2^2x_3^2x_4^2,\ \text{and}$$

$$\Sigma''(x_1^3x_2^2x_3^2x_4x_5+x_1^3x_2^2x_3x_4^2x_5)=\Sigma x_1^3x_2^2x_3^2x_4x_5,$$

$$\tau^4=\Sigma''\{x_1^4x_2^4+4(x_1^4x_2^3x_5+x_1^4x_3x_5^3)+6x_1^4x_2^2x_5^2+8x_1^3x_2^2x_3x_5 \\ +4(x_1^3x_2^2x_3x_5^2+x_1^3x_2^2x_4x_5^3)+4x_1^3x_2^2x_3^2x_4x_5\} \\ +2\Sigma x_1^3x_2^2x_3^2x_4^2+4\Sigma x_1^3x_2^2x_3^2x_4x_5;$$

$$\therefore\ \Sigma\tau^4=3\Sigma x_1^4x_2^4+4\Sigma x_1^4x_2^3x_3+6\Sigma x_1^4x_2^2x_3^2+8\Sigma x_1^3x_2^2x_3x_4 \\ +4\Sigma x_1^3x_2^2x_3^2x_4+12\Sigma x_1^3x_2^2x_3^2x_4^2+36\Sigma x_1^3x_2^2x_3^2x_4x_5.$$

Or, working with U,

$$U^2=\Sigma'(x_1^3x_2x_5+x_1^2x_3x_4)U,$$

developing U, multiplying, reducing, &c., there results

$$U^2=\Sigma'\{(x_1^4x_2^2x_5^2+x_1^4x_3^2x_4^2)+2x_1^4x_2x_3x_4x_5+2(x_1^3x_2^2x_3x_5 \\ +x_1^3x_3^2x_4x_5)+2(x_1^3x_2^2x_3^2x_4+x_1^3x_2^2x_4^2x_5+x_1^3x_2x_3^2x_5^2 \\ +x_1^3x_3x_4^2x_5^2)+2(x_1^3x_2^2x_3^2x_4x_5+x_1^2x_2^2x_3x_4^2x_5)\} \\ =\Sigma'\{(x_1^4x_2^2x_5^2+x_1^4x_3^2x_4^2)+2(x_1^3x_2^2x_3x_5+x_1^3x_3^2x_4x_5)$$

$$+2(x_1^3x_2^3x_3^2x_4+x_1^3x_2^2x_4^2x_5+x_1^2x_2x_3^2x_5^2+x_1^3x_3x_4^2x_5^2)\}$$
$$+2\Sigma x_1^4x_2x_3x_4x_5+2\Sigma x_1^3x_2^2x_3^2x_4x_5;$$
$$\therefore\ \Sigma U^2=2\Sigma x_1^4x_2^2x_3^2+12\Sigma x_1^4x_2x_3x_4x_5+4\Sigma x_1^3x_2^2x_3x_4$$
$$+4\Sigma x_1^3x_2^2x_3^2x_4+12\Sigma x_1^2x_2^2x_3^2x_4x_5.$$

32. By Newton's theorem,

$$nD_n+D_{n-1}\Sigma\theta+\cdots+D_1\Sigma\theta^{n-1}+\Sigma\theta^n=0,$$

whence, making $n=2$, 3, 4 and 5 successively, and reducing by means of (Art. 30)

$$D_1=0, \text{ and } \Sigma\theta=0,$$

we obtain

$$D_2=-\tfrac{1}{2}\Sigma\theta^2,\ D_3=-\tfrac{1}{3}\Sigma\theta^3,\ D_4=-\tfrac{1}{4}(D_2\Sigma\theta^2+\Sigma\theta^4)$$
$$=\tfrac{1}{4}(2D_2^2-\Sigma\theta^4), \text{ and}$$
$$D_5=-\tfrac{1}{5}(D_3\Sigma\theta^2+D_2\Sigma\theta^3+\Sigma\theta^5)=D_2D_3-\tfrac{1}{5}\Sigma\theta^5.$$

33. Applying the same theorem to equation (2), Art. 28, we find

$\Sigma x=0,\ \Sigma x^2=0,\ \Sigma x^3=3\cdot5Q,\ \Sigma x^4=0,\ \Sigma x^5=-5E,$
$\Sigma x^6=3\cdot5^2Q^2,\ \Sigma x^7=0,\ \Sigma x^8=-8\cdot5QE,\ \Sigma x^9=3\cdot5^3Q^3,$
$\Sigma x^{10}=5E^2,\ \Sigma x^{11}=-11\cdot5^2Q^2E,\ \Sigma x^{12}=3\cdot5^4Q^4,$
$\Sigma x^{13}=13\cdot5QE^2,\ \Sigma x^{14}=-14\cdot5^3Q^3E,\ \Sigma x^{15}=3\cdot5^5Q^5-5E^3,$
$\Sigma x^{16}=24\cdot5^2Q^2E^2,\ \Sigma x^{17}=-17\cdot5^4Q^4E,\ \Sigma x^{18}=3\cdot5^6Q^6$
$-18\cdot5QE^3,\ \Sigma x^{19}=38\cdot5^3Q^3E^2,$ and $\Sigma x^{20}=-4\cdot5^6Q^5E+5E^4.$

34. We know that, when (α) and (β) are unequal,

$$\Sigma x_1^\alpha x_2^\beta=\Sigma x^\alpha\Sigma x^\beta-\Sigma x^{\alpha+\beta},$$

and that

$$\Sigma x_1^\alpha x_2^\alpha=\tfrac{1}{2}\{(\Sigma x^\alpha)^2-\Sigma x^{2\alpha}\}.$$

Or, dropping the subject x, and the symbol Σ, we may, after Hirsch, write these relations thus:

$$(\alpha\beta)=[\alpha][\beta]-[\alpha+\beta], \text{ and } (\alpha\alpha)=\tfrac{1}{2}\{[\alpha]^2-[2\alpha]\}.$$

Extending the method commonly employed to establish these relations we find the following:

$$(\alpha\beta\gamma)=[\alpha][\beta][\gamma]-[\alpha+\beta][\gamma]-[\alpha+\gamma][\beta]-[\alpha][\beta+\gamma]$$
$$+2[\alpha+\beta+\gamma],$$
$$(\alpha\beta\beta)=\tfrac{1}{2}\{[\alpha][\beta]^2-2[\alpha+\beta][\beta]-[\alpha][2\beta]+2[\alpha+2\beta]\},$$
$$(\alpha\alpha\alpha)=\tfrac{1}{6}\{[\alpha]^3-3[2\alpha][\alpha]+2[3\alpha]\},$$
$$(\alpha\beta\gamma\delta)=[\alpha][\beta][\gamma][\delta]-[\alpha][\beta][\gamma+\delta]-[\alpha][\beta+\gamma][\delta]$$

$$-[\alpha][\beta+\delta][\gamma]+2[\alpha][\beta+\gamma+\delta]-[\alpha+\beta][\gamma][\delta]$$
$$-[\alpha+\gamma][\beta][\delta]-[\alpha+\delta][\beta][\gamma]+[\alpha+\beta][\gamma+\delta]$$
$$+2[\alpha+\gamma+\delta][\beta]+2[\alpha+\beta+\gamma][\delta]+[\alpha+\delta][\beta+\gamma]$$
$$+2[\alpha+\beta+\delta][\gamma]+[\alpha+\gamma][\beta+\delta]-6[\alpha+\beta+\gamma+\delta],$$

$$(\alpha\beta\gamma\gamma)=\tfrac{1}{2}\{[\alpha][\beta][\gamma]^2-[\alpha][\beta][2\gamma]-2[\alpha][\beta+\gamma][\gamma]$$
$$+2[\alpha][\beta+2\gamma]-[\alpha+\beta][\gamma]^2-2[\alpha+\gamma][\beta][\gamma]$$
$$+[\alpha+\beta][2\gamma]+2[\alpha+2\gamma][\beta]+4[\alpha+\beta+\gamma][\gamma]$$
$$+2[\alpha+\gamma][\beta+\gamma]-6[\alpha+\beta+2\gamma]\},$$

$$(\alpha\alpha\beta\beta)=\tfrac{1}{4}\{[\alpha]^2[\beta]^2-[\alpha]^2[2\beta]-4[\alpha][\alpha+\beta][\beta]$$
$$+4[\alpha][\alpha+2\beta]-[2\alpha][\beta]^2+[2\alpha][2\beta]$$
$$+4[2\alpha+\beta][\beta]+2[\alpha+\beta]^2-6[2\alpha+2\beta]\},$$

$$(\alpha\beta\beta\beta)=\tfrac{1}{6}\{[\alpha][\beta]^3-3[\alpha][\beta][2\beta]+2[\alpha][3\beta]$$
$$-3[\alpha+\beta][\beta]^2+3[\alpha+\beta][2\beta]+6[\alpha+2\beta][\beta]$$
$$-6[\alpha+3\beta]\},\text{ and}$$

$$(\alpha\alpha\alpha\alpha)=\tfrac{1}{24}\{[\alpha]^4-6[\alpha]^2[2\alpha]+8[\alpha][3\alpha]+3[2\alpha]^2$$
$$-6[4\alpha]\}.$$

Hirsch denotes repeating exponents thus:

$\Sigma x_1^\alpha x_2^\alpha$ by (a^2), $\Sigma x_1^\alpha x_2^\alpha x_3^\beta x_4^\beta x_5^\beta$ by $(a^2\beta^3)$, &c.;

but I have thought it conducive to clearness to express them as above. It will be observed that the crotchet [ ] is employed to represent the sum of powers, and the parenthesis ( ) that of other functions of the roots.

By the aid of the formulæ in this article and the values of [1], [2], &c.... [20], given in the last, we may easily calculate the value of any symmetric function of x in terms of Q and E.

35. Thus taking the functions that occur in the value of $\Sigma\tau^4$ (Art. 31), we have *

$$\Sigma(44)=\tfrac{1}{2}\{[4]^2-[8]\}=\tfrac{1}{2}\{0^2-(-8\cdot5\mathrm{QE})\}=4\cdot5\mathrm{QE},$$
$$\Sigma[431]=[4][3][1]-[7][1]-[5][3]-[4]^2+2[8]$$
$$=-5\mathrm{QE},$$
$$\Sigma(422)=\tfrac{1}{2}\{[4][2]^2-2[6][2]-[4]^2+2[8]\}=-8\cdot5\mathrm{QE},$$
$$\Sigma(3311)=\tfrac{1}{4}\{[3]^2[1]^2-[3]^2[2]-4[3][4][1]+4[3][5]$$

* To prevent confusion, the symbol Σ is prefixed to the parenthesis ( ).

$$-[6][1]^2+[6][2]+4[7][1]+2[4]^2-6[8]\}$$
$$=-3\cdot5QE,$$

$$\Sigma(3221)=\tfrac{1}{2}\{[3][1][2]^2-[3][1][4]-2[3]^2[2]+2[3][5]$$
$$-[4][2]^2-2[5][1][2]+[4]^2+2[7][1]+4[6][2]$$
$$+2[5][3]-6[8]\}=-6\cdot5QE,$$

$$\Sigma(2222)=\tfrac{1}{24}\{[2]^4-6[2]^2[4]+8[2][6]+3[4]^2-6[8]\}$$
$$=2\cdot5QE,$$

$$\Sigma(22211)=-E\Sigma(111)=-\frac{E}{6}\{[1]^3-3[2][1]+2[3]\}$$
$$=-5QE;$$

$$\therefore\ \Sigma\tau^4=3\Sigma(44)+4\Sigma(431)+6\Sigma(422))+8\Sigma(3311)$$
$$+4\Sigma(3221)+12\Sigma(2222)+36\Sigma(22211)$$
$$=-4\cdot5^3QE.$$

Also,

$$\Sigma(41111)=-E[3]=-3\cdot5QE,\text{ and}$$
$$\therefore\ \Sigma U^2=2\Sigma(422)+12\Sigma(4111I)+4\Sigma(3311)+4\Sigma(3221)$$
$$+12\Sigma(22211)=-4\cdot5^3QE,$$

which coincides, as it ought to do, with the value of $\Sigma\tau^4$.

36. Now (Art. 32),

$$D_2=-\tfrac{1}{2}\Sigma\theta^2=-\frac{5^2}{2}\Sigma\tau^4=-\frac{5^2}{2}\Sigma U^2,$$

and substituting for $\Sigma\tau^4$ or ΣU^2, its value found in the last article, we obtain

$$D_2=2\cdot5^5QE.$$

37. The expression for τ^4 (Art. 31) may be conveniently written thus:

$$\tau^4=\Sigma'[(44000)+4\{(43001)+(41003)\}+6(42002)$$
$$+8(33101)+4\{(32102)+(32012)\}+4(22211)]$$
$$+2\Sigma(2222)+4\Sigma(22211);$$

or, arranging according to the strict descent of powers, and substituting for the symmetric functions, $\Sigma(2222)$ and $\Sigma(22211)$, their values given in Art. 35,

$$\tau^4=\Sigma'\{(44000)+4(43001)+6(42002)+4(41003)$$
$$+8(33101)+4(32102)+4(32012)+4(22211)\}$$

Moreover (Art. 31),

$$\tau^3=(22000)+2(21001)+(20002)+2(12100)+2(10012)+(02200)+2(01210)+(00220)+2(00121)+(00022).$$

Consequently

$$\tau^6=\Sigma'\{(44000)+4(43001)+6(42002)+4(41003)+8(33101)+4(32102)+4(32012)+4(22211)\}\cdot\{(22000)+2(21001)+(20002)+2(12100)+2(10012)+(02200)+2(00121)+(00022)\}.$$

Developing, reducing,* &c., we are conducted to

$$\tau^6=\Sigma'[(66000)+6\{(65001)+(61005)\}+15\{(64002)+(62004)\}+20(63003)+24(55101)+\{36(54102)+6(54012)+6(52104)+36(52014)\}+24\{(53103)+(53013)\}+\{(44220)+28(44202)+(44022)\}+\{20(44211)+2(44121)+20(44112)\}+8\{(43203)+(43023)\}+40(43113)+\{4(43221)+24(43212)+12(43122)+12(42213)+24(42123)+4(41223)\}+6(42222)+16\{(33321)+(33312)\}+\{52(33222)+8(32322)\}].$$

Applying the theorem in Art. 29 we have

$$\Sigma\tau^6=3\Sigma'(66)+6\Sigma'(651)+15\Sigma(642)+20\Sigma(633)+24\Sigma'(5511)+21\Sigma'(5421)+24\Sigma'(5331)+30\Sigma'(4422)+42\Sigma'(44211)+8\Sigma'(4332)+40\Sigma'(43311)+40\Sigma'(43221)+36\Sigma'(42222)+96\Sigma'(33321)+30\Sigma'(33222).$$

By the aid of the formulæ in Art. 34, and the values of [1], [2], &c. in Art. 33, we find

$$\Sigma'(66)=3\cdot5^4Q^4,\ \Sigma(651)=-3\cdot5^4Q^4,\ \Sigma(642)=-3\cdot5^4Q^4$$
$$\Sigma'(633)=3\cdot5^4Q^4,\ \Sigma(5511)=0,\ \Sigma(5421)=0,\ \Sigma(5331)=0,$$
$$\Sigma'(4422)=0,\ \Sigma(44211)=0,\ \Sigma(4332)=0,\ \Sigma(43311)=0,$$

* It may be convenient here to illustrate the principle of reduction proceeded upon in this Paper in dealing with circular functions. According to the notation of the text, represent $\Sigma' x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta x_5^\epsilon$ by $\Sigma'(\alpha\beta\gamma\delta\epsilon)$; and suppose (*e.g.*) that, of the five exponents, γ is the greatest; then $\Sigma'(\alpha\beta\gamma\delta\epsilon)$ must be replaced by its equivalent $\Sigma'(\gamma\delta\epsilon\alpha\beta)$. Or, suppose that the greatest exponent (γ) is repeated, and the function takes the form $\Sigma'(\alpha\gamma\gamma\delta\epsilon)$; this must be replaced by its equivalent $\Sigma'(\gamma\gamma\delta\epsilon\alpha)$. And so on. This mode of reduction is uniformly followed in the text. *The greatest exponent leads.*

$\Sigma'(43221)=0$, $\Sigma(42222)=0$, $\Sigma(33321)=0$, and $\Sigma(33222)=0$.

Hence

$$\Sigma\tau^6=6\cdot5^4Q^4.$$

38. Again (Art. 31):—

$$\begin{aligned}U^2=\Sigma'[&\{(42002)+(40220)\}+2\{(33101)+(30311)\}\\&+2\{(32210)+(32021)+(31202)+(30122)\}]\\&+2\Sigma(41111)+2\Sigma(22211);\end{aligned}$$

or since (Art. 35)

$$\Sigma(41111)=-3\cdot5QE, \text{ and } \Sigma(22211)=-5QE,$$

$$\begin{aligned}U^2=\Sigma'\{&(42002)+(40220)+2(33101)+2(32210)\\&+2(32021)+2(31202)+2(30311)+2(30122)\}\\&-8\cdot5QE.\end{aligned}$$

Next,

$$\begin{aligned}U=\Sigma'\{&(21001)+(20110)\}=(21001)+(20110)\\&+(12100)+(11020)+(10201)+(10012)\\&+(02011)+(01210)+(01102)+(00121)\}.\end{aligned}$$

Now,

$$\begin{aligned}U^3=\Sigma'\{&(42002)+(40220)+2(33101)+2(32210)\\&+2(32021)+2(31202)+2(30311)+2(30122)\}U\\&-8\cdot5QEU.\end{aligned}$$

Hence if, in the function affected by Σ', we substitute for U its value, develope, reduce, &c., as in the last article, we shall obtain

$$\begin{aligned}U^3=\Sigma'[&\{(63003)+(60330)\}+\{(62112)+(61221)\}\\&+3\{(54102)+(52014)+(51240)+(50421)\}\\&+3\{(53022)+(52320)+52203)+(50232)\}\\&+4\{(53211)+(52131)+(51312)+(51123)\}\\&+3\{(44310)+(44013)+(41430)+(41403)\}\\&+6\{(44121)+(42411)\}+6\{(43302)+(43230)\\&+(42033)+(40323)\}+4\{(43113)+(41331)\}\\&+\{(43212)+2(43122)+2(42321)+(42231)\\&+2(42213)+(42123)+(41322)+2(41232)\}\\&+8(42222)+4\{(33321)+(33312)+(33231)\\&+(33132)\}+2\{(33222)+(32322)\}]-8\cdot5QEU;\end{aligned}$$

$$\begin{aligned}\therefore\ \Sigma U^3 = {} & 2\Sigma(633)+2\Sigma(62211)+3\Sigma(5421)+6\Sigma(5322)\\ & +8\Sigma(53211)+6\Sigma(4431)+12\Sigma(44211)\\ & +12\Sigma(4332)+8\Sigma(43311)+6\Sigma(43221)\\ & +48\Sigma(42222)+24\Sigma(33321)+12\Sigma(33222)\\ & -8\cdot 5\mathrm{Q}\mathrm{E}\Sigma\mathrm{U}.\end{aligned}$$

But $\Sigma(62211)=0$, $\Sigma(5322)=0$, $\Sigma(53211)=0$, $\Sigma(4431)=0$, and $\Sigma U=0$; hence making these substitutions, and also for the other symmetric functions whose values are given in the last article, we obtain

$$\Sigma U^3 = 6\cdot 5^4 Q^4,$$

which confirms the value of $\Sigma\tau^6$.

39. By Art. 32,

$$D_3 = -\tfrac{1}{3}\Sigma\theta^3 = \frac{5^3}{3}\Sigma\tau^6 = \frac{5^3}{3}\Sigma U^3;$$

and substituting for $\Sigma\tau^6$ or ΣU^3 its value (Art. 37 and 38), we have

$$D_3 = 2\cdot 5^7 Q^4.$$

40. Since $\Sigma'(42222) = \Sigma(42222) = E^2[2] = 0$, and $\Sigma''\{(33222) + (32322)\} = \Sigma(33222) = E^3\Sigma(11) = 0$, the value of τ^6 (Art. 37) may be expressed thus:

$$\begin{aligned}\tau^6 = \Sigma'[& (66000)+6\{(65001)+(61005)\}+15\{64002)\\ & +(62004)\}+20(63003)+24(55101)+\{36(54102)\\ & +6(54012)+6(52104)+36(52014)\}+24\{(53103)\\ & +(53013)\}+\{(44220)+28(44202)+(44022)\}\\ & +\{20(44211)+2(44121)+20(44112\}+8\{(43203)\\ & +(43023)\}+40(43113)+\{4(43221)+24(43212)\\ & +12(43122)+12(42213)+24(42123)+4(41223)\}\\ & +16\{(33321)+(33312)\}+44(33222)].\end{aligned}$$

We might now arrange these functions according to descending powers, multiply by τ^3, and proceed precisely as in Art. 37, but it will be more convenient to work with U.

41. Since $\Sigma''(42222)=0$, $\Sigma''\{(33321)+(33312)+(33231)+(33132)\}=\Sigma'(33321)=0$, and $\Sigma''\{(33222)+(32322)\}=\Sigma(33222)=0$, we have (Art. 38)

$$U^3 = \Sigma''\{(63003)+(62112)+(61221)+(60330)+3(54102)$$
$$+4(53211)+3(53022)+3(52320)+3(52203)$$
$$+4(52131)+3(52014)+4(51312)+3(51240)$$
$$+4(51123)+3(50421)+3(50232)+3(44310)$$
$$+6(44121)+3(44013)+6(43302)+6(43230)$$
$$+(43212)+2(43122)+4(43113)+6(42411)$$
$$+2(42321)+(42231)+2(42213)+(42123)$$
$$+6(42033)+3(41430)+3(41403)+4(41331)$$
$$+(41322)+2(41232)+6(40323)\} - 8\cdot 5QEU.$$

Multiplying by U, and proceeding as in Art. 38, we find, after making the necessary reductions, &c.,

$$U^4 = \Sigma''[\{(84004)+(80440)\}+2\{(83113)+(81331)\}$$
$$+2(82222)+4\{(75103)+(73015)+(71350)$$
$$+(70531)\}+4\{(74023)+(73204)+(72430)$$
$$+(70342)\}+8\{(74212)+(72241)+(72124)$$
$$+(71422)\}+8\{(73321)+(73132)+(72313)$$
$$+(71233)\}+6\{(66202)+(60622)\}+4\{(65014)$$
$$+(64105)+(61540)+(60451)\}+10\{(65311)$$
$$+(63151)+(61513)+(61135)\}+12\{(65122)$$
$$+(62521)+(62215)+(61252)\}+6\{(64420)$$
$$+(64042)+(62404)+(60244)\}+10\{(64114)$$
$$+(61441)\}+12\{(64303)+(63340)+(63034)$$
$$+(60433)\}+\{20(64231)+2(64213)+6(64123)$$
$$+20(63412)+6(63214)+2(63124)+6(62431)$$
$$+2(62341)+20(62143)+2(61432)+6(61342)$$
$$+20(61324)\}+\{4(63322)+4(63232)+18(63223)$$
$$+18(62332)+4(62323)+4(62233)\}+12\{(55402)$$
$$+(55204)+(50542)+(50524)\}+12\{(55141)$$
$$+(54511)\}+12\{(55330)+(55033)+(53530)$$
$$+(53503)\}+8\{(55312)+(55213)+(51532)$$
$$+(51523)\}+12\{(55222)+(52522)\}+\{16(54421)$$
$$+6(54214)+16(54142)+6(54124)+6(52441)$$
$$+16(52414)+6(51442)+16(51244)\}+\{6(54331)$$
$$+12(54313)+16(54133)+16(53431)+6(53413)$$
$$+12(53341)+16(53314)+6(53143)+12(53134)$$

$$+12(51433)+16(51343)+6(51334)\}+\{16(54322)$$
$$+4(54232)+4(54223)+4(53422)+16(53242)$$
$$+4(53224)+4(52432)+16(52423)+4(52342)$$
$$+4(52324)+4(52243)+16(52234)\}+6\{(53332)$$
$$+(53323)+(53233)+(52333)\}+24(44440)$$
$$+4\{(44431)+(44413)+(44341)+(44143)\}$$
$$+12\{(44422)+(44242)\}+\{10(44332)+8(44323)$$
$$+10(44233)+10(43432)+10(43423)+8(42433)\}$$
$$+4(43333)]-8\cdot5QEU^3.$$

Applying the theorem in Art. 29 we have,

$$\Sigma U^4=2\Sigma(844)+4\Sigma(83311)+12\Sigma(82222)+4\Sigma(7531)$$
$$+4\Sigma(7432)+16\Sigma(74221)+16\Sigma(73321)$$
$$+12\Sigma(6622)+4\Sigma(6541)+20\Sigma(65311)$$
$$+24\Sigma(65221)+12\Sigma(6442)+20\Sigma(64411)$$
$$+24\Sigma(6433)+28\Sigma(64321)+52\Sigma(63322)$$
$$+24\Sigma(5542)+24\Sigma(55411)+48\Sigma(5533)$$
$$+16\Sigma(55321)+72\Sigma(55222)+44\Sigma(54421)$$
$$+68\Sigma(54331)+48\Sigma(54322)+36\Sigma(53332)$$
$$+144\Sigma(4444)+24\Sigma(44431)+72\Sigma(44422)$$
$$+56\Sigma(44332)+24\Sigma(43333)-8\cdot5QE\Sigma U^3.$$

But

$\Sigma(844)=-8\cdot5^2Q^2E^2$, $\Sigma(83311)=11\cdot5^2Q^2E^2$, $\Sigma(82222)=3\cdot5^3Q^2E^2$, $\Sigma(7531)=3\cdot5^3Q^2E^2$, $\Sigma(7432)=4\cdot5^2Q^2E^2$, $\Sigma(74221)=6\cdot5^3Q^2E^2$, $\Sigma(73321)=6\cdot5^2Q^2E^2$, $\Sigma(6622)=11\cdot5^3Q^2E^2$, $\Sigma(6541)=-9\cdot5^3Q^2E^2$, $\Sigma(65311)=7\cdot5^3Q^2E^2$, $\Sigma(65221)=-3\cdot5^3Q^2E^2$, $\Sigma(6442)=-10\cdot5^2Q^2E^2$, $\Sigma(64411)=2\cdot5^3Q^2E^2$, $\Sigma(6433)=6\cdot5^2Q^2E^2$, $\Sigma(64321)=6\cdot5^2Q^2E^2$, $\Sigma(63322)=3\cdot5^3Q^2E^2$, $\Sigma(5542)=8\cdot5^2Q^2E^2$, $\Sigma(55411)=-5^3Q^2E^2$, $\Sigma(5533)=0$, $\Sigma(55321)=-6\cdot5^2Q^2E^2$, $\Sigma(55222)=3\cdot5^3Q^2E^2$, $\Sigma(54421)=0$, $\Sigma(54331)=6\cdot5^2Q^2E^2$, $\Sigma(54322)=-3\cdot5^3Q^2E^2$, $\Sigma(53332)=0$, $\Sigma(4444)=2\cdot5^2Q^2E^2$, $\Sigma(44431)=-2\cdot5^3Q^2E^2$, $\Sigma(44422)=5^2Q^2E^2$, $\Sigma(44332)=0$, $\Sigma(43333)=0$, and (Art. 35) $\Sigma U^3=-4\cdot5^3QE$.

Consequently,

$$\Sigma U^4=4\cdot5^6Q^2E^2;$$

and, therefore, (Art. 32)

$$D_4=\tfrac{1}{4}(2D_2^2-\Sigma\theta^4)=\tfrac{1}{4}(2D_2^2-5^4\Sigma U^4)=\tfrac{1}{4}(2D_2^2-4\cdot5^{10}Q^2E^2);$$

or, since (Art. 36)

$$D_2=2\cdot5^5QE,$$

$$D_4=5^{10}Q^2E^2.$$

42. Since $\Sigma''(82222)=\Sigma(82222)=3\cdot5^2Q^2E^2$, $\Sigma''\{(63322)+(63232)+(63223)+(62332)+(62323)+(62233)\}=\Sigma(63322)=3\cdot5^2Q^2E^2$, $\Sigma''\{(55222)+(52522)\}=\Sigma(55222)=3\cdot5^2Q^2E^2$, $\Sigma''\{(54331)+(54313)+(54133)+(53431)+(53413)+(53341)+(53314)+(53143)+(53134)+(51433)+(51343)+(51334)\}=\Sigma(54331)=6\cdot5^2Q^2E^2$, $\Sigma''\{(54322)+(54232)+(54223)+(53422)+(53242)+(53224)+(52432)+(52423)+(52342)+(52324)+(52243)+(52234)\}=\Sigma(54322)=-3\cdot5^2Q^2E^2$, $\Sigma''\{(53332)+(53323)+(53233)+(52333)\}=\Sigma(53332)=0$, $\Sigma''(44440)=\Sigma(4444)=2\cdot5^2Q^2E^2$, $\Sigma''\{(44431)+(44413)+(44341)+(44143)\}=\Sigma(44431)=-2\cdot5^2Q^2E^2$, $\Sigma''\{(44422)+(44242)\}=\Sigma(44422)=5^2Q^2E^2$, $\Sigma''\{(44332)+(44323)+(44233)+(43432)+(43423)+(42433)\}=\Sigma(44332)=0$, and $\Sigma''(43333)=\Sigma(43333)=0$, we have by substitution in the value of U^4 (Art. 41),

$$\begin{aligned}U^4=\Sigma'[&\{(84004)+(80440)\}+2\{(83113)+(81331)\}\\&+4\{(75103)+(73015)+(71350)+(70531)\}\\&+4\{(74023)+(73204)+(72430)+(70342)\}\\&+8\{(74212)+(72241)+(72124)+(71422)\}\\&+8\{(73321)+(73132)+(72313)+(71233)\}\\&+6\{(66202)+(60622)\}+4\{(65014)+(64105)\\&+(61540)+(60451)\}+10\{(65311)+(6315I)\\&+(61513)+(61135)\}+12\{(65122)+(62521)\\&+(62215)+(61252)\}+6\{(64420)+(64042)\\&+(62404)+(60244)\}+10\{(64114)+(61441)\}\\&+12\{(64303)+(63340)+(63034)+(60433)\}\\&+\{20(64231)+2(64213)+6(64123)+20(63412)\\&+6(63214)+2(63124)+6(62431)+2(62341)\\&+20(62143)+2(61432)+6(61342)+20(61324)\}\end{aligned}$$

$$
\begin{aligned}
&+14\{(63223)+(62332)\}+12\{(55402)+(55204)\\
&+(50542)+(50524)\}+12\{(55141)+(54511)\}\\
&+12\{(55330)+(55033)+(53530)+(53503)\}\\
&+8\{(55312)+(55213)+(51532)+(51523)\}\\
&+\{16(54421)+6(54214)+16(54142)+6(54124)\\
&+6(52441)+16(52414)+6(51442)+16(51244)\}\\
&+\{6(54313)+10(54133)+10(53431)+6(53341)\\
&+10(53314)+6(53134)+6(51433)+10(51343)\}\\
&+12\{(54322)+(53242)+(52423)+(52234)\}\\
&+2\{(44332)+(44233)+(43432)+(43423)\}]\\
&+26\cdot5^3Q^3E^3-8\cdot5QEU^3;
\end{aligned}
$$

or, arranging the terms affected by Σ' according to the descent of powers,

$$
\begin{aligned}
U^4=\Sigma'\{&(84004)+2(83113)+2(81331)+(80440)\\
&+4(75103)+8(74212)+4(74023)+8(73321)\\
&+4(73204)+8(73132)+4(73015)+4(72430)\\
&+8(72313)+8(72241)+8(72124)+8(71422)\\
&+4(71350)+8(71233)+4(70531)+4(70342)\\
&+6(66202)+10(65311)+12(65122)+4(65014)\\
&+6(64420)+12(64303)+20(64231)+2(64213)\\
&+6(64123)+10(64114)+4(64105)+6(64042)\\
&+20(63412)+12(63340)+14(63223)+6(63214)\\
&+10(63151)+2(63124)+12(63034)+12(62521)\\
&+6(62431)+6(62404)+2(62341)+14(62332)\\
&+12(62215)+20(62143)+4(61540)+10(61513)\\
&+10(61441)+2(61432)+6(61342)+20(61324)\\
&+12(61252)+10(61135)+6(60622)+4(60451)\\
&+12(60433)+6(60244)+12(55402)+12(55330)\\
&+8(55312)+8(55213)+12(55204)+12(55141)\\
&+12(55033)+12(54511)+16(54421)+12(54322)\\
&+6(54313)+6(54214)+16(54142)+10(54133)\\
&+6(54124)+12(53530)+12(53503)+10(53431)\\
&+6(53341)+10(53314)+12(53242)+6(53134)\\
&+6(52441)+12(52423)+16(52414)+12(52234)\\
&+8(51532)+8(51523)+6(51442)+6(51433)
\end{aligned}
$$

$$+10(51343)+16(51244)+12(50542)+12(50524)$$
$$+2(44332)+2(44233)+2(43432)+2(43423)\}$$
$$+26\cdot5^3Q^2E^2-8\cdot5QEU^2.$$

Multiplying by U, and proceeding as before, we obtain

$$U^5=[\{(\overline{10}5005)+(\overline{10}0550)\}+3\{(\overline{10}4114)+(\overline{10}1441)\}$$
$$+2\{(\overline{10}3223)+(\overline{10}2332)\}+5\{(96104)+(94016)$$
$$+(91460)+(90641)\}+5\{(95024)+(94205)$$
$$+(92540)+(90452)\}+14\{(95213)+(93125)$$
$$+(92351)+(91532)\}+14\{(94133)+(93431)$$
$$+(93314)+(91343)\}+16\{(94322)+(93242)$$
$$+(92423)+(92234)\}+10\{(87203)+(83027)$$
$$+(82370)+(80732)\}+5\{(86015)+(85106)$$
$$+(81650)+(80561)\}+\{24(86312)+20(86123)$$
$$+24(83261)+20(83216)+20(82631)+24(82136)$$
$$+24(81623)+20(81362)\}+18\{(85115)+(81551)\}$$
$$+\{20(85304)+10(85043)+10(84530)+20(84035)$$
$$+20(83450)+10(83405)+20(80543)+10(80354)\}$$
$$+\{24(85421)+3(85214)+12(85124)+12(84215)$$
$$+24(84152)+3(84125)+12(82541)+24(82514)$$
$$+3(82451)+3(81542)+12(81452)+24(81245)\}$$
$$+48\{(85232)+(83522)+(82325)+(82253)\}$$
$$+48\{(84413)+(84341)+(83144)+(81434)\}$$
$$+48\{(84224)+(82442)\}+\{4(84323)+6(84233)$$
$$+6(83432)+4(83342)+6(83324)+4(83234)$$
$$+4(82433)+6(82343)\}+60(83333)+18\{(77411)$$
$$+(77114)+(71741)+(71714)\}+30\{(77222)$$
$$+(72722)\}+\{10(76520)+20(76205)+10(75062)$$
$$+20(75026)+10(72605)+20(72560)+20(70652)$$
$$+10(70256)\}+\{30(76403)+20(76034)+30(74360)$$
$$+20(74306)+20(73640)+30(73046)+30(70634)$$
$$+20(70463)\}+\{24(76214)+30(76142)+30(74621)$$
$$+24(74126)+24(72461)+30(72416)+24(71642)$$
$$+30(71264)\}+\{54(76331)+14(76313)+54(73613)$$
$$+14(73361)+54(73163)+14(73136)+14(71633)$$
$$+54(71336)\}+24\{(76223)+(73226)+(72632)$$

$$+(72362)\}+\{54(75512)+54(75251)+14(75215)$$
$$+14(75125)+14(72551)+54(72155)+14(71552)$$
$$+54(71525)\}+30\{(75440)+(74504)+(74054)$$
$$+(70445)\}+\{32(75314)+42(75143)+44(75134)$$
$$+42(74531)+44(74315)+32(74135)+44(73541)$$
$$+32(73451)+42(73415)+32(71543)+44(71453)$$
$$+42(71354)\}+\{32(75422)+6(75224)+32(74252)$$
$$+6(74225)+6(72542)+32(72524)+6(72452)$$
$$+32(72245)\}+\{8(75332)+68(75323)+6(75233)$$
$$+6(73532)+8(73523)+68(73352)+6(73325)$$
$$+8(73253)+68(73235)+68(72533)+6(72353)$$
$$+8(72335)\}+6\{(74441)+(74414)+(74144)$$
$$+(71444)\}+\{82(74432)+14(74423)+14(74342)$$
$$+20(74324)+82(74243)+20(74234)+20(73442)$$
$$+82(73424)+14(73244)+20(72443)+14(72434)$$
$$+82(72344)\}+8\{(74333)+(73433)+(73343)$$
$$+(73334)\}+20\{(66611)+(66161)\}+30\{(66350)$$
$$+(66053)+(65630)+(65603)\}+48\{(66521)$$
$$+(66125)+(62651)+(62615)\}+\{12(66431)$$
$$+34(66413)+34(66314)+12(66134)+12(63641)$$
$$+12(63614)+34(61643)+34(61634)\}$$
$$+\{48(66422)+72(66242)+48(66224)+72(64622)$$
$$+48(62642)+48(62624)\}+\{30(66332)+6(66323)$$
$$+30(66233)+30(63632)+30(63623)+6(62633)\}$$
$$+36\{(65405)+(65045)+(64550)+(60554)\}$$
$$+\{40(65531)+36(65513)+36(65351)+6(65315)$$
$$+40(65153)+6(65135)+6(63551)+40(63515)$$
$$+36(63155)+6(61553)+36(61535)+40(61355)\}$$
$$+\{12(65522)+12(65252)+40(65225)+40(62552)$$
$$+12(62525)+12(62255)\}+\{52(65441)+30(65414)$$
$$+12(65144)+12(64541)+52(64514)+30(64451)$$
$$+12(64415)+52(64154)+30(64145)+30(61544)$$
$$+12(61454)+52(61445)\}+\{10(65432)+28(65423)$$
$$+48(65342)+18(65324)+12(65243)+56(65234)$$
$$+12(64532)+48(64523)+28(64352)+56(64325)$$

$$
\begin{aligned}
&+10(64253)+18(64235)+56(63542)+10(63524)\\
&+18(63452)+12(63425)+48(63254)+28(63245)\\
&+18(62543)+28(62534)+56(62453)+48(62435)\\
&+12(62354)+10(62345)\}+46\{(65333)+(63533)\\
&+(63353)+(63335)\}+20\{(64442)+(64424)\\
&+(64244)+(62444)\}+\{32(64433)+32(64343)\\
&+48(64334)+48(63443)+32(63434)+32(63344)\}\\
&+18\{(55541)+(55514)+(55451)+(55154)\}\\
&+58\{(55532)+(55523)+(55352)+(55253)\}\\
&+\{26(55442)+40(55424)+26(55244)+26(54542)\\
&+26(54524)+40(52544)\}+\{14(55433)+24(55343)\\
&+14(55334)+24(54533)+14(53543)+14(53534)\}\\
&+24\{(54443)+(54434)+(54344)+(53444)\}]\\
&+26\cdot5^3Q^2E^2U-8\cdot5QEU^3.
\end{aligned}
$$

The bar placed over the number 10 indicates that it is a single exponent, so that $\Sigma'(\overline{10}5005)$ represents $\Sigma' x_1^{10}x_2^5x_5^5$, $\Sigma'(\overline{10}4114)$ represents $\Sigma' x_1^{10}x_2^4x_3x_4x_5^4$, &c. Whence by the theorem in Art. 29,

$$
\begin{aligned}
\Sigma U^5=&2\Sigma(\overline{10}55)+6\Sigma(\overline{10}4411)+4\Sigma(\overline{10}3322)+5\Sigma(9641)\\
&+5\Sigma(9542)+14\Sigma(95321)+28\Sigma(94331)\\
&+32\Sigma(94322)+10\Sigma(8732)+5\Sigma(8651)\\
&+44\Sigma(86321)+36\Sigma(85511)+30\Sigma(8543)\\
&+39\Sigma(85421)+96\Sigma(85322)+96\Sigma(84431)\\
&+96\Sigma(84422)+20\Sigma(84332)+360\Sigma(83333)\\
&+72\Sigma(77411)+180\Sigma(77222)+30\Sigma(7652)\\
&+50\Sigma(7643)+54\Sigma(76421)+136\Sigma(76331)\\
&+48\Sigma(76322)+136\Sigma(75521)+60\Sigma(7544)\\
&+118\Sigma(75431)+76\Sigma(75422)+164\Sigma(75332)\\
&+36\Sigma(74441)+232\Sigma(74432)+48\Sigma(74333)\\
&+120\Sigma(66611)+60\Sigma(6653)+96\Sigma(66521)\\
&+92\Sigma(66431)+336\Sigma(66422)+132\Sigma(66332)\\
&+72\Sigma(6554)+164\Sigma(65531)+128\Sigma(65522)\\
&+188\Sigma(65441)+172\Sigma(65432)+276\Sigma(65333)\\
&+120\Sigma(64442)+224\Sigma(64433)+108\Sigma(55541)\\
&+348\Sigma(55532)+184\Sigma(55442)+104\Sigma(55433)
\end{aligned}
$$

$$+144\Sigma(54443)+26\cdot5^3Q^2E^2\Sigma U-8\cdot5QE\Sigma U^3.$$

But

$\Sigma(\bar{1}055)=-5^6Q^5E+6\cdot5E^4$, $\Sigma(\bar{1}04411)=-3\cdot5^5Q^5E+5E^4$, $\Sigma(\bar{1}03322)=5E^4$, $\Sigma(9641)=4\cdot5E^4$, $\Sigma(9524)=-6\cdot5^5Q^5E+4\cdot5E^4$, $\Sigma(95321)=-5E^4$, $\Sigma(94331)=2\cdot5E^4$, $\Sigma(94322)=2\cdot5E^4$, $\Sigma(8732)=-6\cdot5^5Q^5E+4\cdot5E^4$, $\Sigma(8651)=4\cdot5E^4$, $\Sigma(86321)=4\cdot5E^4$, $\Sigma(85511)=-3\cdot5^5Q^5E+5E^4$, $\Sigma(8543)=4\cdot5E^4$, $\Sigma(85421)=-5E^4$, $\Sigma(85322)=2\cdot5E^4$, $\Sigma(84431)=2\cdot5E^4$, $\Sigma(84422)=5E^4$, $\Sigma(84332)=-3\cdot5E^4$, $\Sigma(83333)=5E^4$, $\Sigma(77411)=-3\cdot5^5Q^5E+5E^4$, $\Sigma(77222)=2\cdot5E^4$, $\Sigma(7652)=-6\cdot5^5Q^5E+4\cdot5E^4$, $\Sigma(7643)=-5E^4$, $\Sigma(76421)=4\cdot5E^4$, $\Sigma(76331)=2\cdot5E^4$, $\Sigma(76322)=-3\cdot5E^4$, $\Sigma(75521)=2\cdot5E^4$, $\Sigma(7544)=2\cdot5E^4$, $\Sigma(75431)=-5E^4$, $\Sigma(75422)=-3\cdot5E^4$, $\Sigma(75332)=2\cdot5E^4$, $\Sigma(74441)=-5E^4$, $\Sigma(74432)=2\cdot5E^4$, $\Sigma(74333)=-5E^4$, $\Sigma(66611)=-5^5Q^5E+2\cdot5E^4$, $\Sigma(6653)=2\cdot5E^4$, $\Sigma(66521)=-3\cdot5E^4$, $\Sigma(66431)=-3\cdot5E^4$, $\Sigma(66422)=5E^4$, $\Sigma(66332)=5E^4$, $\Sigma(6554)=-3\cdot5E^4$, $\Sigma(65531)=2\cdot5E^4$, $\Sigma(65522)=5E^4$, $\Sigma(65441)=2\cdot5E^4$, $\Sigma(65432)=-5E^4$, $\Sigma(65333)=-5E^4$, $\Sigma(64442)=-5E^4$, $\Sigma(64433)=5E^4$, $\Sigma(55541)=-5E^4$, $\Sigma(55532)=-5E^4$, $\Sigma(55442)=5E^4$, $\Sigma(55433)=5E^4$, $\Sigma(54443)=-5E^4$, $\Sigma U=0$, and $\Sigma U^3=6\cdot5^4Q^4$. Hence, by substitution, we have

$$\Sigma U^5=-158\cdot5^6Q^5E+5^6E^4;\ \therefore\ \Sigma\theta^5=-5^5\Sigma U^5$$
$$=158\cdot5^{11}Q^5E-5^{11}E^4.$$

Now (Art. 32),

$$D_5=D_2D_3-\tfrac{1}{5}\Sigma\theta^5,$$

and (Art. 36 and 37)

$$D_2=2\cdot5^5QE,\text{ and }D_3=2\cdot5^7Q^4;$$

consequently

$$D_5=-58\cdot5^{10}Q^5E+5^{10}E^4.$$

43. Again: $\Sigma'(83333)=\Sigma(83333)=5E^4$, $\Sigma'\{(77222)+(72722)\}=\Sigma(77222)=2\cdot5E^4$, $\Sigma'\{(75332)+(75323)+(75233)+(73532)+(73523)+(73352)+(73325)+(73253)+(73235)+(72533)+(72353)+(72335)\}$

$=\Sigma(75332)=2\cdot5E^4$, $\Sigma'\{(74441)+(74414)+(74144)$
$+(71444)\}=\Sigma(74441)=-5E^4$, $\Sigma''\{(74432)+(74423)$
$+(74342)+(74324)+(74243)+(74234)+(73442)$
$+(73424)+(73244)+(72443)+(72434)+(72344)\}$
$=\Sigma(74432)=2\cdot5E^4$, $\Sigma'\{(74333)+(73433)+(73343)$
$+(73334)\}=\Sigma(74333)=-5E^4$, $\Sigma'\{(66611)+(66161)\}$
$\Sigma(66611)=-5^5Q^5E+2\cdot5E^4$, $\Sigma''\{(66422)+(66242)$
$+(66224)+(64622)+(62642)+(62624)\}=\Sigma(66422)$
$=5E^4$, $\Sigma''\{66332)+(66323)+(66233)+(63632)+(63623)$
$+(62633)\}=\Sigma(66332)=5E^4$, $\Sigma''\{(65531)+(65513)$
$+(65351)+(65315)+(65153)+(65135)+(63551)$
$+(63515)+(63155)+(61553)+(61535)+(61355)\}$
$=\Sigma(65531)=2\cdot5E^4$, $\Sigma''\{(65522)+(65252)+(65225)$
$+(62552)+(62525)+(62255)\}=\Sigma(65522)=5E^4$,
$\Sigma''\{(65441)+(65414)+(65144)+(64541)+(64514)$
$+(64451)+(64415)+(64154)+(64145)+(61544)$
$+(61454)+(61445)\}=\Sigma(65441)=2\cdot5E^4$, $\Sigma''\{(65432)$
$+(65423)+(65342)+(65324)+(65243)+(65234)$
$+(64532)+(64523)+(64352)+(64325)+(64253)$
$+(64235)+(63542)+(63524)+(63452)+(63425)$
$+(63254)+(63245)+(62543)+(62534)+(62453)$
$+(62435)+(62354)+(62345)\}=\Sigma(65432)=-5E^4$,
$\Sigma''\{(65333)+(63533)+(63353)+(63335)\}=\Sigma(65333)$
$=-5E^4$, $\Sigma''\{(64442)+(64424)+(64244)+(62444)\}$
$=\Sigma(64442)=-5E^4$, $\Sigma''\{(64433)+(64343)+(64334)$
$+(63443)+(63434)+(63344)\}=\Sigma(64433)=5E^4$,
$\Sigma''\{(55541)+(55514)+(55451)+(55154)\}=\Sigma(55541)$
$=-5E^4$, $\Sigma''\{(55532)+(55523)+(55352)+(55253)\}$
$=\Sigma(55532)=-5E^4$, $\Sigma''\{(55442)+(55424)+(55244)$
$+(54542)+(54524)+(52544)\}=\Sigma(55442)=5E^4$,
$\Sigma''\{(55433)+(55343)+(55334)+(54533)+(53543)$
$+(53534)\}=\Sigma(55433)=5E^4$, and $\Sigma''\{(54443)+(54434)$
$+(54344)+(53444)\}=\Sigma(54443)=-5E^4$. Hence, by substitution in the value of U^5 given in the last article, we have

$$
\begin{aligned}
U^5 = \Sigma' [& \{(\overline{10}5005) + (\overline{10}0550)\} + 3\{(\overline{10}4114) + (\overline{10}1441)\} \\
& + 2\{(\overline{10}3223) + (\overline{10}2332)\} + 5\{(96104) + (94016) \\
& + (91460) + (90641)\} + 5\{(95024) + (94205) \\
& + (92540) + (90452)\} + 14\{(95213) + (93125) \\
& + (92351) + (91532)\} + 14\{(94133) + (93431) \\
& + (93314) + (91343)\} + 16\{(94322) + (93242) \\
& + (92423) + (92234)\} + 10\{(87203) + (83027) \\
& + (82370) + (80732)\} + 5\{(86015) + (85106) \\
& + (81650) + (80561)\} + \{24(86312) + 20(86123) \\
& + 24(83261) + 20(83216) + 20(82631) + 24(82136) \\
& + 24(81623) + 20(81362)\} + 18\{(85115) + (81551)\} \\
& + \{20(85304) + 10(85043) + 10(84530) + 20(84035) \\
& + 20(83450) + 10(83405) + 20(80543) + 10(80354)\} \\
& + \{24(85421) + 3(85214) + 12(85124) + 12(84215) \\
& + 24(84152) + 3(84125) + 12(82541) + 24(82514) \\
& + 3(82451) + 3(81542) + 12(81452) + 24(81245)\} \\
& + 48\{(85232) + (83522) + (82325) + (82253)\} \\
& + 48\{(84413) + (84341) + (83144) + (81434)\} \\
& + 48\{(84224) + (82442)\} + \{4(84323) + 6(84233) \\
& + 6(83432) + 4(83342) + 6(83324) + 4(83234) \\
& + 4(82433) + 6(82343)\} + 18\{(77411) + (77114) \\
& + (71741) + (71714)\} + \{10(76520) + 20(76205) \\
& + 10(75062) + 20(75026) + 10(72605) + 20(72560) \\
& + 20(70652) + 10(70256)\} + \{30(76403) + 20(76034) \\
& + 30(74360) + 20(74306) + 20(73640) + 30(73046) \\
& + 30(70634) + 20(70463)\} + \{24(76214) + 30(76142) \\
& + 30(74621) + 24(74126) + 24(72461) + 30(72416) \\
& + 24(71642) + 30(71264)\} + \{54(76331) + 14(76313) \\
& + 54(73613) + 14(73361) + 54(73163) + 14(73136) \\
& + 14(71633) + 54(71336)\} + 24\{(76223) + (73226) \\
& + (72632) + (72362)\} + \{54(75512) + 54(75251) \\
& + 14(75215) + 14(75125) + 14(72551) + 54(72155) \\
& + 14(71552) + 54(71525)\} + 30\{(75440) + (74504) \\
& + (74054) + (70445)\} + \{32(75314) + 42(75143) \\
& + 44(75134) + 42(74531) + 44(74315) + 32(74135)
\end{aligned}
$$

$$+44(73541)+32(73451)+42(73415)+32(71543)$$
$$+44(71453)+42(71354)\}+\{32(75422)+6(75224)$$
$$+32(74252)+6(74225)+6(72542)+32(72524)$$
$$+6(72452)+32(72245)\}+\{2(75332)+62(75323)$$
$$+2(73523)+62(73352)+2(73253)+62(73235)$$
$$+62(72533)+2(72335)\}+\{68(74432)+6(74324)$$
$$+68(74243)+6(74234)+6(73442)+68(73424)$$
$$+6(72443)+68(72344)\}+30\{(66350)+(66053)$$
$$+(65630)+(65603)\}+48\{(66521)+(66125)$$
$$+(62651)+(62615)\}+\{12(66431)+34(66413)$$
$$+34(66314)+12(66134)+12(63641)+12(63614)$$
$$+34(61643)+34(61634)\}+24\{(66242)+(64622)\}$$
$$+24\{(66332)+(66233)+(63632)+(63623)\}$$
$$+36\{(65405)+(65045)+(64550)+(60554)\}$$
$$+\{34(65531)+30(65513)+30(65351)+34(65153)$$
$$+34(63515)+30(63155)+30(61535)+34(61355)\}$$
$$+28\{(65225)+(62552)\}+\{40(65441)+18(65414)$$
$$+40(64514)+18(64451)+40(64154)+18(64145)$$
$$+18(61544)+40(61445)\}+\{18(65423)+38(65342)$$
$$+8(65324)+2(65243)+46(65234)+2(64532)$$
$$+38(64523)+18(64352)+46(64325)+8(64235)$$
$$+46(63542)+8(63452)+2(63425)+38(63254)$$
$$+18(63245)+8(62543)+18(62534)+46(62453)$$
$$+38(62435)+2(62354)\}+16\{(64334)+(63443)\}$$
$$+14\{(55424)+(52544)\}+10\{(55343)+(54533)\}]$$
$$-4\cdot5^6Q^5E+184\cdot5E^4+26\cdot5^3Q^2E^2U-8\cdot5QEU^3;$$

or, arranging according to descending powers,

$$U^5=\Sigma''\{(\bar{1}05005)+3(\bar{1}04114)+2(\bar{1}03223)+2(\bar{1}02332)$$
$$+3(\bar{1}01441)+(\bar{1}00550)+5(96104)+14(95213)$$
$$+5(95024)+16(94322)+5(94205)+14(94133)$$
$$+5(94016)+14(93431)+14(93314)+16(93242)$$
$$+14(93125)+5(92540)+16(92423)+14(92351)$$
$$+16(92234)+14(91532)+5(91460)+14(91343)$$
$$+5(90641)+5(90452)+10(87203)+24(86312)$$
$$+20(86123)+5(86015)+24(85421)+20(85304)$$

$$
\begin{aligned}
&+48(85232)+3(85214)+12(85124)+18(85115)\\
&+5(85106)+10(85043)+10(84530)+48(84413)\\
&+48(84341)+4(84323)+6(84233)+48(84224)\\
&+12(84215)+24(84152)+3(84125)+20(84035)\\
&+48(83522)+20(83450)+6(83432)+10(83405)\\
&+4(83342)+6(83324)+24(83261)+4(83234)\\
&+20(83216)+48(83144)+10(83027)+20(82631)\\
&+12(82541)+24(82514)+3(82451)+48(82442)\\
&+4(82433)+10(82370)+6(82343)+48(82325)\\
&+48(82253)+24(82136)+5(81650)+24(81623)\\
&+18(81551)+3(81542)+12(81452)+48(81434)\\
&+20(81362)+24(81245)+10(80732)+5(80561)\\
&+20(80543)+10(80354)+18(77411)+18(77114)\\
&+10(76520)+30(76403)+54(76331)+14(76313)\\
&+24(76223)+24(76214)+20(76205)+30(76142)\\
&+20(76034)+54(75512)+30(75440)+32(75422)\\
&+2(75332)+62(75323)+32(75314)+54(75251)\\
&+6(75224)+14(75215)+42(75143)+44(75134)\\
&+14(75125)+10(75062)+20(75026)+30(74621)\\
&+42(74531)+30(74504)+68(74432)+30(74360)\\
&+6(74324)+44(74315)+20(74306)+32(74252)\\
&+68(74243)+6(74234)+6(74225)+32(74135)\\
&+24(74126)+30(74054)+20(73640)+54(73613)\\
&+44(73541)+2(73523)+32(73451)+6(73442)\\
&+68(73424)+42(73415)+14(73361)+62(73352)\\
&+2(73253)+62(73235)+24(73226)+54(73163)\\
&+14(73136)+30(73046)+24(72632)+10(72605)\\
&+20(72560)+14(72551)+6(72542)+62(72533)\\
&+32(72524)+24(72461)+6(72452)+6(72443)\\
&+30(72416)+24(72362)+68(72344)+2(72335)\\
&+32(72245)+54(72155)+18(71741)+18(71714)\\
&+24(71642)+14(71633)+14(71552)+32(71543)\\
&+54(71525)+44(71453)+42(71354)+54(71336)\\
&+30(71264)+20(70652)+30(70634)+20(70463)\\
&+30(70445)+10(70256)+48(66521)+12(66431)
\end{aligned}
$$

$$+34(66413)+30(66350)+24(66332)+34(66314)$$
$$+24(66242)+24(66233)+12(66134)+48(66125)$$
$$+30(66053)+30(65630)+30(65603)+34(65531)$$
$$+30(65513)+40(65441)+18(65423)+18(65414)$$
$$+36(65405)+30(65351)+38(65342)+8(65324)$$
$$+2(65243)+46(65234)+28(65225)+34(65153)$$
$$+36(65045)+24(64622)+36(64550)+2(64532)$$
$$+38(64523)+40(64514)+18(64451)+18(64352)$$
$$+16(64334)+46(64325)+8(64235)+40(64154)$$
$$+18(64145)+12(63641)+24(63632)+24(63623)$$
$$+12(63614)+46(63542)+34(63515)+8(63452)$$
$$+16(63443)+2(63425)+38(63254)+18(63245)$$
$$+30(63155)+48(62651)+48(62615)+28(62552)$$
$$+8(62543)+18(62534)+46(62453)+38(62435)$$
$$+2(62354)+34(61643)+34(61634)+18(61544)$$
$$+30(61535)+40(61445)+34(61355)+36(60554)$$
$$+14(55424)+10(55343)+10(54533)+14(52544)\}$$
$$-4\cdot5^6Q^5E+184\cdot5E^4+26\cdot5^3Q^3E^2U-8\cdot5QEU^3.$$

If now we were to multiply by U, and proceed as before, we should obtain the value of U^6; thence we might pass (Art. 29) to ΣU^6; and D_6 would be given by (Newton's theorem)

$$6D_6=2D_2^3+6D_2D_4+3D_3^2-\Sigma\theta^6;\text{ or}$$
$$D_6=5^6(2\cdot5^3Q^3+\frac{14\cdot5^9}{3}Q^3E^3-\Sigma U^6).$$

But it is not necessary to calculate this coefficient, since it has been already found (see Sec. II., Art. 27) that

$$D_6=\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6=5^{14}Q^3.$$

44. The equation in θ is, therefore,

$$\theta^6+2\cdot5^5QE\theta^4+2\cdot5^7Q^4\theta^3+5^{10}Q^2E^2\theta^2-5^{10}(58Q^5-E^3)E\theta$$
$$+5^{14}Q^3=0,$$

which coincides in every point with the result obtained by Mr. Cockle. (See § 12 of his "Researches in the Higher Algebra," published in the present volume.)

45. The foregoing calculations are laborious, but they

have been conducted with the greatest care, and verified in a variety of ways. I believe that the equation in θ will be found to have an important bearing on the theory of quintics; and therefore I have thought it desirable to place its accuracy beyond all reasonable question.

46. We have seen how steadily the Method of Symmetric Products mounts up to the higher equations, and in the course of our investigations we have been conducted to most significant results. Guided therefore by the analogy afforded by the lower equations, we might next consider the application of the method to quintics. But my professional occupations will not permit me at present to enter into this interesting discussion. I may however remark that even if the method fail to achieve the solution of the general problem, it will probably help to settle a controversy in which mathematicians of the greatest eminence have taken opposite sides, and to throw light upon the question, respecting which so much has been written, of the possibility or impossibility of expressing a root of the general equation of the fifth degree by a finite combination of radicals and rational functions.

ADDENDUM.

The resolvent product (θ) for a quintic equation is immediately connected with the last coefficient of the equation of the fourth degree which occurs in Lagrange's theory, the coefficient in question being by the present theory determined as the root of an equation of the sixth degree. And, since it is part and parcel of Lagrange's theory, that when any one root of the equation is known (*ex. gr.* if there is a root zero, which is the case considered in the second section), the roots of the proposed quintic can be determined by radicals, the form of the results

obtained in the second section may be considered as given *á priori* by Lagrange's theory.

Lagrange arrives at a reducing equation which may be written

$$\vartheta^4 - T\vartheta^3 + U\vartheta^2 - X\vartheta + Y = 0,$$

where $\vartheta = \{f(i^m)\}^5$, and the coefficients T, U, &c. are, each of them, susceptible of only six different values, and Y is obviously the fifth power of the expression denoted in the above Paper by θ. Now if, in seeking these coefficients, we commence with T, we shall be conducted to functions of five dimensions with respect to x. Thus, developing

$$\vartheta = \{f(i)\}^5 = (x_1 + i\,x_2 + i^2 x_3 + i^3 x_4 + i^4 x_5)^5$$

according to powers of i, we have

$$\vartheta = \xi_1 + \xi_2 i + \xi_3 i^2 + \xi_4 i^3 + \xi_5 i^4,$$

where, employing (for convenience) the new cyclical symbol,

$$\begin{aligned}
\xi_1 &= \Sigma x^5 + 10\Sigma'\{2x_1^3(x_2x_5 + x_3x_4) + 3x_1(x_2^2x_5^2 + x_3^2x_4^2)\} \\
&\quad + 120x_1x_2x_3x_4x_5, \\
\xi_2 &= 5\Sigma'(x_1^4x_2 + 2x_1^3x_4^2 + 4x_1^3x_3x_5 + 6x_1^2x_2^2x_5 + 12x_1^2x_2x_3x_4), \\
\xi_3 &= 5\Sigma'(x_1^4x_3 + 2x_1^3x_2^2 + 4x_1^3x_4x_5 + 6x_1^2x_2x_4^2 + 12x_1^2x_2x_3x_5), \\
\xi_4 &= 5\Sigma'(x_1^4x_4 + 2x_1^3x_5^2 + 4x_1^3x_2x_3 + 6x_1^2x_3^2x_5 + 12x_1^2x_2x_4x_5), \\
\xi_5 &= 5\Sigma'(x_1^4x_5 + 2x_1^3x_3^2 + 4x_1^3x_2x_4 + 6x_1^2x_2^2x_3 + 12x_1^2x_3x_4x_5);
\end{aligned}$$

and if we distinguish this value of ϑ by $\vartheta(i)$, the others will be denoted by $\vartheta(i^2)$, $\vartheta(i^3)$ and $\vartheta(i^4)$. Consequently

$$T = \Sigma\vartheta = 4\xi_1 + (\xi_2 + \xi_3 + \xi_4 + \xi_5)(i + i^2 + i^3 + i^4) = 5\xi_1 - \Sigma\xi;$$

or, since

$$\begin{aligned}
\Sigma\xi &= \Sigma x^5 + 5\Sigma x_1^4x_2 + 10\Sigma x_1^3x_2^2 + 20\Sigma x_1^3x_2x_3 + 30\Sigma x_1^2x_2^2x_3 \\
&\quad + 60\Sigma x_1^2x_2x_3x_4 + 120x_1x_2x_3x_4x_5, \\
T &= 4\Sigma x^5 - 5\Sigma x_1^4x_2 - 10\Sigma x_1^3x_2^2 - 20\Sigma x_1^3x_2x_3 - 30\Sigma x_1^2x_2^2x_3 \\
&\quad - 60\Sigma x_1^2x_2x_3x_4 + 50\Sigma'\{2x_1^3(x_2x_5 + x_3x_4) + 3x_1(x_2^2x_5^2 \\
&\quad + x_3^2x_4^2)\} + 480x_1x_2x_3x_4x_5.
\end{aligned}$$

Applying the Σ' process (Sec. II., Art. 19),

$$\begin{aligned}
\Sigma' x_1^3(x_2x_5 + x_3x_4) &= \Sigma'\{x_1^2(x_2x_5 + x_3x_4)(\Sigma x - x_2 - x_3 - x_4 - x_5)\} \\
&= \Sigma x\,\Sigma' x_1^2(x_2x_5 + x_3x_4) - \Sigma' x_1^2(x_2^2x_3 + x_2^2x_5 + x_3^2x_4 + x_3^2x_5 \\
&\quad + x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5)
\end{aligned}$$

$$= \Sigma x \Sigma' x_1^3 (x_2 x_5 + x_3 x_4) - \Sigma x_1^3 x_2^3 x_3 + \Sigma' x_1 (x_2^3 x_5^3 + x_3^3 x_4^3) - \Sigma x_1^2 x_2 x_3 x_4 .$$

Consequently

$$T = 4\Sigma x^5 - 5\Sigma x_1^4 x_2 - 10\Sigma x_1^3 x_2^2 - 20\Sigma x_1^3 x_2 x_3 - 130\Sigma x_1^2 x_2^2 x_3 - 160\Sigma x_1^2 x_2 x_3 x_4 + 480 x_1 x_2 x_3 x_4 x_5 + 100\Sigma x \Sigma' x_1^2 (x_2 x_5 + x_3 x_4) + 250\Sigma' x_1 (x_2^2 x_5^2 + x_3^2 x_4^2).$$

But by commencing the calculations with Y (in place of T) we should have been conducted to the fifth power of the four-dimensioned function discussed in the above Essay. It is the form of that function which has enabled me to introduce τ with such effect into the calculations; and it is the introduction of τ which has enabled Mr. Cockle (*Phil. Mag.*, May 1859) to give a simpler form of root than the expression

$$\tfrac{1}{5}\left\{A_1 + i^m X + i^{2m} X_2 + i^{3m} X_3 + \frac{\theta}{i^m X X_2 X_3}\right\}$$

to which he would otherwise have had recourse.

The advantage of Lagrange's and Vandermonde's process is that the solution of their sextics would lead directly to that of the quintic. The disadvantages are the high dimensions of the symmetric functions, and the consequent difficulties of calculation and verification. The Method of Symmetric Products, considered independently, leads only indirectly to the solution of a quintic. But it has the advantage of lower dimensioned functions and of greater facilities for calculation and verification, $\Sigma' x_1^2 (x_2 x_5 + x_3 x_4)$ being a more manageable function than $\Sigma' x_1^3 (x_2 x_5 + x_3 x_4)$ or $\Sigma' x_1 (x_2^2 x_5^2 + x_3^2 x_4^2)$. The unsymmetric expression for the roots of a quintic given by Lagrange and Vandermonde is undoubtedly an element of the theory of quintics as treated in the foregoing Essay, but the progress made will be tested by the simplicity of the results to which the Method of Symmetric Products has conducted us. If the difficulties of dealing with symmetric functions increase in more than arithmetical proportion to their dimensions, it

is a moderate expression to say that, by making *products* instead of *powers* of linear functions of the roots the basis of the theory, the difficulties of the calculation are diminished by one-fifth. The cyclical symbol (Σ'), again, combining five terms in one, diminishes by four-fifths the four-fifths that remain. So that altogether more than five-sixths of the labour disappears. And since there is no six-valued function of five letters of less than four dimensions, it is improbable that any view of the subject can lead to a simpler reducing sextic than that in θ.

XV. — *On the Partitions and Reticulations of the r-gon.*

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1. Although I have given the formulæ of an inductive method (*Philosophical Transactions*, 1857, "On the partitions of the r-gon and the r-ace"), whereby the partitions of the r-gon made by any k diagonals, none crossing another, may be enumerated, much remains to be done before they are expressed directly in terms of r and k. The case of $k=r-3$, that of the triangular partitions, has been discussed in a previous part of the present volume (p. 43), in which this direct expression has been investigated. The most convenient classification seems to be that according to the number of marginal faces; it is, at least, essential in the theory of the polyedra.

In this Paper will be given the required expressions of the partitions of the r-gon by $k+1$ diagonals, in which there are two marginal faces.

2. Such a partition, F, will always have $2+x$ angles unoccupied by a diagonal. Let x of these — all except one in each marginal face — vanish, that is, become angles of 180°. F has now become F′, a $(k+2)$-partitioned r'-gon $(r'=r-x)$, having two marginal triangles, and no angle, besides their two vertices, unoccupied by a diagonal. No face, either of F or of F′, has more than two diagonals in

its circuit; for if any face has three, sections along them will cut off three polygons having each at least one angle outside its section unoccupied by a diagonal, because every polygon has at least one such angle above its base. But F and F′ have each only two such angles; wherefore no face of either has more than two diagonals, and consequently no face of F′ is ampler than a quadrilateral.

3. If we pare away all the margin of F′, all its faces, except the two marginal triangles, will become quadrilaterals. F′ is now F″, a $(2k+4)$-gon, having k quadrilaterals, two triangles, and $k+1$ diagonals. Let all the k consecutive edges ab, bc, cd .... of quadrilaterals vanish except y, and in the y remaining quadrilaterals let all the opposite edges $a'b'$, $b'c'$, $c'd'$ .... vanish except z; there are now only z quadrilaterals. This can be done in

$$\frac{k^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{z\,|\,-1}}{1^{z\,|\,1}} \text{ different ways.}$$

In order that F′ should be one of the resulting figures, it is necessary that the spared edges, together with the four in the marginal triangles, should be r'. That is,

$$k+z=r'-4, \text{ or}$$
$$z=r'-k-4.$$

Wherefore F′ is one among

$$\Sigma_y \frac{k^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{r'-k-4\,|\,-1}}{1^{r'-k-4\,|\,1}} \text{ figures, } (y \gtrless 0).$$

4. To find this sum, let $r'-k-4=\theta$; then all terms vanish in which $y<\theta$, and the series is

$$\frac{k^{\theta\,|\,-1}}{1^{\theta\,|\,1}} \cdot \frac{\theta^{\theta\,|\,-1}}{1^{\theta\,|\,1}} + \frac{k^{\theta+1\,|\,-1}}{1^{\theta+1\,|\,1}} \cdot \frac{(\theta+1)^{\theta\,|\,-1}}{1^{\theta\,|\,1}} + \frac{k^{\theta+2\,|\,-1}}{1^{\theta+2\,|\,1}} \cdot \frac{(\theta+2)^{\theta\,|\,-1}}{1^{\theta\,|\,1}} + \cdots$$
$$= \frac{k^{\theta\,|\,-1}}{1^{\theta\,|\,1}} + \frac{k^{\theta+1\,|\,-1}}{1^{\theta+1\,|\,1}} \cdot \frac{\theta+1}{1} + \frac{k^{\theta+2\,|\,-1}}{1^{\theta+2\,|\,1}} \cdot \frac{(\theta+2)^{2\,|\,-1}}{1^{2\,|\,1}}$$
$$+ \frac{k^{\theta+3\,|\,-1}}{1^{\theta+3\,|\,1}} \cdot \frac{(\theta+3)^{3\,|\,-1}}{1^{3\,|\,1}} + \cdots$$
$$= \frac{k^{\theta\,|\,-1}}{1^{\theta\,|\,1}} \left\{ 1 + \frac{k-\theta}{\theta+1} \cdot \frac{\theta+1}{1} + \frac{(k-\theta)(k-\theta-1)}{(\theta+1)(\theta+2)} \cdot \frac{(\theta+1)(\theta+2)}{1 \cdot 2} + \cdots \right\}$$

$$= \frac{k^{\theta\,|\,-1}}{1^{\theta\,|\,1}} \cdot 2^{k-\theta} = \frac{k^{r'-k-4\,|\,-1}}{1^{r'-k-4\,|\,1}} \cdot 2^{2k-r'+4}; \text{ or}$$

$$\Sigma_y \frac{k^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{r'-k-4\,|\,-1}}{1^{r'-k-4\,|\,1}} = \frac{k^{r'-k-4\,|\,-1}}{1^{r'-k-4\,|\,1}} \cdot 2^{2k-r'+4},$$

is the number of figures F′, which all differ at least in their position, all being placed so as to have their two marginal triangles on either hand.

5. The r-gon F is made from the r'-gon F′ by depositing x points ($r'+x=r$) in any one or more of $r'-2$ positions, viz., about the vertices of the two marginal triangles of F′, and on the $r'-4$ other edges of F′. This can be done in

$$\frac{(x+1)^{r-3\,|\,1}}{1^{r-3\,|\,1}} = \frac{r-3^{r-s-3\,|\,-1}}{1^{r-s-3\,|\,1}} \text{ different ways;}$$

and the product of this number into Σ_y just found, taken for every value of $x=r-r'$, is the entire number of r-gons F. That is,* if $R^2(r, k+1)$ denote the number of these r-gons which have $k+1$ diagonals and two axes of reversion, $R(r, k+1)$ that of the r-gons F which have one axis of reversion, and $I^2(r, k+1)$ that of those which are *doubly irreversible*, that is, which have no axis of reversion, but have a sequence of configuration repeated in the circuit; and if $I(r, k+1)$ denote the number of these F which have no symmetry, but are *irreversible*, *i.e.* have their lower face different from their upper, we obtain, by writing $r-x$ for r',

* An example of the class $R^2(r,2)$ is formed by placing two triangles on opposite sides either of a square or of a hexagon. An example of $R^{ag}(r,2)$ is got by placing two triangles about a hexagon, so that one side of it shall stand between them. One of $R^{ai}(r,2)$ is made by placing two triangles about a hexagon, so that no side shall be between them. Two triangles placed on any two sides of a pentagon form an example of $R^{mo}(r,2)$; and if a diameter be drawn which is not one of the two axes in any of the class $R^2(r,2)$, it becomes one of $I^2(r,3)$. — Vid. Art. (8, 9, 10, 11). The symmetry of these figures is fully discussed in the memoir referred to in the first article, in the *Philosophical Transactions*.

$$\{R^2+2(R+I^2)+4I\}(r,k+1)$$
$$=\Sigma_x \frac{k^{r-x-k-4\,|\,-1}}{1^{r-x-k-4\,|\,1}} \cdot \frac{(r-3)^{r-x-3\,|\,-1}}{1^{r-x-3\,|\,1}} \cdot 2^{2k-r+x+4};$$

where x has every value that gives a result, namely,

$$x \lessdot 0,\ x \lessdot r-2k-4,\ x \gtrdot r-k-4.$$

For each one of the doubly reversible will be once only constructed, having but one possible posture about either axis; each one of the R will be twice made, once with either extremity of its axis uppermost; each of I^2 will also be twice constructed, to show both surfaces; and each of the I will be four times made, with either marginal face on the right hand, and with either surface uppermost.

I cannot find that this sum Σ_x has any simpler expression than the finite series which it represents.

By this formula, when $R^2(r,k+1)$, $R(r,k+1)$ and $I^2(r,k+1)$ have been otherwise determined, we easily obtain the number of the largest class, the class $I(r,k+1)$ of these $(k+2)$-partitions of the r-gon which have two marginal faces.

6. Let the r-gon E be one of the number $R^2(r,k+1)$. This E has a certain number $x+2$ of angles unoccupied by a diagonal; and, by the vanishing of x of these, E becomes the r'-gon E′ ($r'=r-x$), having two marginal triangles. As the x points were symmetrically placed with respect to two axes of symmetry, their disappearance has not disturbed the symmetry, and E′ has the same two axes of reversion. All the $k+1$ diagonals of E′ are parallel to one and perpendicular to the other of these axes, and E′ is made up of k quadrilaterals and two marginal triangles, and $r'=2k+4$.

When k is odd, $r'=4i+6$, and there is a central quadrilateral, on each side of which stand $\frac{1}{2}(k-1)$ others. We turn E′ into E by adding x points in the circuit so as to preserve the symmetry about the two axes. When $x=4j+2$, two of these will be placed at opposite mid-

edges of the central quadrilateral, and the rest symmetrically about the two axes on the four half-edges of it, and on $4i+4$ other edges of E′. That is, $\frac{1}{4}(x-2)$ points are to be laid in one or more groups in $i+2$ positions in the fourth part of the circuit of E′, which can be done in

$$\frac{\left(\frac{x+2}{4}\right)^{i+1\,|\,1}}{1^{i+2\,|\,1}} = \frac{\left(\frac{r-2k-2}{4}\right)^{\frac{2k+2}{4}|1}}{1^{\frac{2k+2}{4}|1}} \text{ ways.}$$

When $x=4j$, there are no points at the mid-edges of the central quadrilateral, but $\frac{1}{4}x$ points are deposited in the $i+2$ positions before mentioned, in some one of

$$\frac{\left(\frac{x+4}{4}\right)^{\frac{r-2}{4}|1}}{1^{\frac{r-2}{4}|1}} = \frac{\left(\frac{r-2k}{4}\right)^{\frac{2k+2}{4}|1}}{1^{\frac{2k+2}{4}|1}} \text{ ways.}$$

When k is even, $r'=4i+4$, there is no central quadrilateral, and $x=4j$. In this case j points are deposited on one-fourth part of E′ on $i+1=\frac{1}{4}r'$ positions, in some one of

$$\frac{\left(\frac{x+4}{4}\right)^{\frac{r'-4}{4}|1}}{1^{\frac{r'-4}{4}|1}} = \frac{\left(\frac{r-2k}{4}\right)^{\frac{k}{2}|1}}{1^{\frac{k}{2}|1}} \text{ ways.}$$

The number R^2 of r-gons E that can be thus constructed on the r'-gon E′, to have two axes of reversion, may be thus expressed,

$$R^2 = \frac{\left(\frac{r-2k-2}{4}\right)^{\frac{k+1}{2}|1}}{1^{\frac{k+1}{2}|1}} + \frac{\left(\frac{r-2k}{4}\right)^{\frac{k+1}{2}|1}}{1^{\frac{k+1}{2}|1}} + \frac{\left(\frac{r-2k}{4}\right)^{\frac{k}{2}|1}}{1^{\frac{k}{2}|1}},$$

if we define that every irreducible fraction containing r or k shall be accounted zero. Of these three terms in R^2 either two or all will always thus vanish.

7. We take up next an r-gon of the number $R(r, k+1)$, having one axis of reversion, two marginal faces, and $k+1$

diagonals. This axis, if it lies, of course an undrawn diameter, through two mid-edges of a central face, is an *agonal* axis; if it is drawn or drawable through two angles of G, it is a *diagonal* axis; if it lies, of course undrawn, through one angle and an opposite mid-edge, it is a *monogonal* axis.

8. Let the r-gon G have an agonal axis of reversion. By the erasure, as before, of x angles not occupied by a diagonal, G becomes the r'-gon G', $(r'=r-x)$, having the same axis, and no unoccupied angle except the two vertices of the marginal triangles; and G' becomes the $(2k+4)$-gon G'', if we pare its margin, having $k=2i+1$ quadrilaterals and two triangles. Of these k, let all the consecutive edges vanish except y, and in these y quadrilaterals let all the opposite edges vanish except z, whereby the k quadrilaterals are reduced to z. This can be done in

$$\frac{i^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{z\,|\,-1}}{1^{z\,|\,1}} = \frac{(\tfrac{1}{2}(k-1))^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{z\,|\,-1}}{1^{z\,|\,1}} \text{ ways,}$$

so that the reduction of G'' shall be effected symmetrically on both sides of the agonal axis. One of our results will be G', if the spared edges

$$i+z=\tfrac{1}{2}(r'-6), \text{ or}$$
$$z=\tfrac{1}{2}(r'-k-5);$$

wherefore the number of figures obtained from G'' is

$$\Sigma_y \frac{(\tfrac{1}{2}(k-1))^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{\frac{1}{2}(r'-k-5)\,|\,-1}}{1^{\frac{1}{2}(r'-k-5)\,|\,1}}; \ (y \geqq 0).$$

This series is easily proved, by the reasoning of Art. 4, to be

$$S=\frac{(\tfrac{1}{2}(k-1))^{\frac{1}{2}(r'-k-5)\,|\,-1}}{1^{\frac{1}{2}(r'-k-5)\,|\,1}} 2^{k-\frac{1}{2}r'+2}$$

which is the number of r'-gons G', to one of which we reduced G. This one becomes G by the addition of $\frac{1}{2}x$ points in $\frac{1}{2}r'$ positions, viz., about the vertex of a marginal triangle, on two opposite half-edges of the central quadrilateral, and on $\frac{1}{2}(r'-6)$ other edges of G' all on the same

side of the axis, and by the repetition of the operation in order reversed on the other side of it.

The number of figures G thus obtained from G′ is

$$\frac{(\frac{1}{2}x+1)^{(\frac{1}{2}r'-1)\,|\,1}}{1^{\frac{1}{2}(r'-1)\,|\,1}} = \frac{(\frac{1}{2}(r'+x)-1)^{(\frac{1}{2}r'-1)\,|\,-1}}{1^{\frac{1}{2}(r'-1)\,|\,1}},$$

and the product of this number into S just found is the entire number of r-gons G that have an agonal axis of reversion placed in every position about that axis. Among them will be once constructed every doubly reversible R^2 which has an agonal axis perpendicular to the axis through its marginal faces, and every singly reversible R^{ag} twice, once with either extremity of its axis upwards. This product then, summed for all values of $x=r-r'$, is

$$(R^2+2R)(r, k+1),$$

wherefore putting for R^2 its value from Art. 6, when k is odd, and $r-2k=4m$,

$$R^{ag}(r, k+1)$$
$$=\tfrac{1}{2}\Sigma_x\left\{\frac{(\frac{1}{2}r-1)^{\frac{1}{2}(r-x)-1\,|\,-1}}{1^{\frac{1}{2}(r-x)-1\,|\,1}}\cdot\frac{(\frac{1}{2}(k-1))^{\frac{1}{2}(r-x-k-5)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-5)\,|\,1}}\cdot 2^{k-\frac{1}{2}(r-x)+2}\right.$$
$$\left.-\frac{(\frac{1}{4}(r-2k))^{\frac{k+1}{2}|1}}{1^{\frac{k+1}{2}|1}}\right\},$$

where $x\lessdot 0$, $x\lessdot r-2k-4$, $x\gtrdot r-k-5$.

The fractions here, as before, are to be reckoned zeros, if they are not integers. This requires that k shall be odd, and r and x both even.

9. The number of $R^{mo}(r, k+1)$ of r-gons K having two marginal faces and a monogonal axis is next to be enumerated. If between the two edges forming the angle through which the axis passes we introduce an edge, the r-gon K becomes an $(r+1)$-gon K′ having an agonal axis, and the figure is either one of $R^2(r+1, k+1)$, where k is odd and $r+1=2k+4m$, or one of $R^{ag}(r+1, k+1)$. And each of $R^2(r+1, k+1)$ will give one K, by the vanishing of an edge carrying the agonal axis which does not

pass through the marginal faces, for this axis has the same configuration at both extremities; while each of $R^{ag}(r+1, k+1)$ will give two figures K, by the vanishing of an edge carrying the agonal axis, for this axis has different configurations at its two extremities. That is, $(r+1=2k+4m)$

$$R^{mo}(r, k+1) = R^{2}(r+1, k+1) + 2R^{ag}(r+1, k+1).$$

We have just found that for $r=2k+4m$

$$(R^{2}+2R^{ag})(r, k+1)$$

$$= \Sigma_x \frac{(\frac{1}{2}r-1)^{\frac{1}{2}(r-x)-1\,|\,-1}}{1^{\frac{1}{2}(r-x)-1\,|\,1}} \cdot \frac{(\frac{1}{2}(k-1))^{\frac{1}{2}(r-x-k-5)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-5)\,|\,1}} \cdot 2^{k-\frac{1}{2}(r-x)+2}$$

$$= R^{mo}(r-1, k+1);$$

wherefore

$$R^{mo}(r, k+1)$$

$$= \Sigma_x \frac{(\frac{1}{2}(r-1))^{\frac{1}{2}(r-x-1)\,|\,-1}}{1^{\frac{1}{2}(r-x-1)\,|\,1}} \cdot \frac{(\frac{1}{2}(k-1))^{\frac{1}{2}(r-x-k-4)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-4)\,|\,1}} \cdot 2^{k-\frac{1}{2}(r-x-3)}$$

where $x \not< 0$, $x \not< r-2k-3$, $x \not> r-k-4$, and fractions irreducible are counted zeros. That is, k is odd, r is odd, and x is even.

10. We are next to enumerate the r-gons $R^{di}(r, k+1)$, which have $k+1$ diagonals, a diagonal axis, and two marginal faces, which axis may be either drawn or undrawn.

Let H be one of these figures having a drawn diagonal axis. It will have $x+2$ angles not occupied by a diagonal, by the vanishing of which H becomes H′ an r'-gon $(r'=r-x)$ having the same axis and two marginal triangles. By removal of the margin, H′ becomes H″, a $(2k+4)$-gon having k quadrilaterals and two marginal triangles. Of the $\frac{1}{2}k$ edges of these quadrilaterals that make one-fourth of the circuit of H″, excluding the triangles, let all but y vanish, and in these y quadrilaterals let all the opposite edges vanish except z. The same changes being made in the other half of H″, so as to maintain the symmetry about the axis, H′ will be one of the resulting figures, if the spared edges

$$\tfrac{1}{2}k+z=\tfrac{1}{2}(r'-4), \text{ or}$$
$$z=\tfrac{1}{2}(r'-k-4).$$

The number of figures H′ to which H″ is thus reducible is

$$\Sigma_y \frac{(\frac{1}{2}k)^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{x\,|\,-1}}{1^{x\,|\,1}} = \Sigma_y \frac{(\frac{1}{2}k)^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{\frac{1}{2}(r'-k-4)\,|\,-1}}{1^{\frac{1}{2}(r'-k-4)\,|\,1}},$$

a sum which is shown by the reasoning of Art. 4 to be

$$S=\frac{(\frac{1}{2}k)^{\frac{1}{2}(r'-k-4)\,|\,-1}}{1^{\frac{1}{2}(r'-k-4)\,|\,1}} \cdot 2^{k-\frac{1}{2}r'+2}.$$

Any one of these S r'-gons H′ becomes an r-gon H, if $\frac{1}{2}x$ points be placed in any of $(\frac{1}{2}r'-1)$ positions on half the r'-gon, namely about the vertex of a marginal triangle, and on $\frac{1}{2}(r'-4)$ other edges of H′ on the same side of the axis, and if the operations be repeated in reverse order on the other side of it. That is, every figure H′ gives

$$\frac{(\frac{1}{2}x+1)^{\frac{1}{2}(r'-2)\,|\,2}}{1^{\frac{1}{2}(r'-2)\,|\,1}} = \frac{(\frac{1}{2}(r'+x-2))^{\frac{1}{2}(r'-2)\,|\,-1}}{1^{\frac{1}{2}(r'-2)\,|\,1}} \text{ figures H.}$$

The product of this number into S, summed for all values of $x=r-r'$, is the entire number of r-gons having $k+1$ diagonals and reversible about a drawn diagonal axis in every position about it. In other words, this product is, putting $R^{di}(r, k+1)'$ for those having one axis only,

$$2R^{di}(r, k+1)'+R^{2}(r, k+1),$$

where for R^2 is to be put its value when k is even, and $r-2k=4m$; for each of R^{di} has two configurations about the axis, and each of R^2 has only one about the axis which does not pass through the marginal faces. Wherefore, putting $r'=r-x$, we obtain

$$R^{di}(r, k+1)'$$
$$\tfrac{1}{2}\Sigma_x \left\{ \frac{(\frac{1}{2}(r-2))^{\frac{1}{2}(r-x-2)\,|\,-1}}{1^{\frac{1}{2}(r-x-2)\,|\,1}} \cdot \frac{(\frac{1}{2}k)^{\frac{1}{2}(r-x-k-4)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-4)\,|\,1}} \cdot 2^{k-\frac{1}{2}(r-x)+2} \right.$$
$$\left. - \frac{(\frac{1}{2}(r-2k))^{\frac{k}{2}\,|\,1}}{1^{\frac{k}{2}\,|\,1}} \right\},$$

where $x \lessdot 0$, $x \lessdot r-2k-4$, $x \gtrdot r-k-4$, and irreducible

fractions are to be accounted zeros. Here r, k and x are all even.

11. If we erase the diagonal axis in H, we obtain an r-gon H_1, having the same two marginal faces, k diagonals, and an undrawn diagonal axis. If $R^{di}(r,k)''$ denote the number of such figures H_1, we have

$$R^{di}(r,k)''=R^{di}(r,k+1)', \text{ and}$$
$$R^{di}(r,k+1)''=R^{di}(r,k+2)',$$

which differs from $R^{di}(r,k+1)'$ only in having $k+1$ for k. That is,

$$R^{di}(r,k+1)''$$
$$=\Sigma_x{}'\frac{1}{2}\left\{\frac{(\frac{1}{2}(r-2))^{\frac{1}{2}(r-x-2)\,|\,-1}}{1^{\frac{1}{2}(r-x-2)\,|\,1}}\cdot\frac{(\frac{1}{2}(k+1))^{\frac{1}{2}(r-x-k-5)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-5\,|\,1}}\cdot 2^{k-\frac{1}{2}(r-x)+3}\right.$$
$$\left.-\frac{(\frac{1}{4}(r-2k-2))^{\frac{k+1}{2}|1}}{1^{\frac{k+1}{2}|1}}\right\},$$

in which x has every even value, so that

$$x\lessdot 0,\ x\lessdot(r-2k-6),\ x\gtrdot(r-k-5).$$

Here k is odd, r and x are both even, and, as before, irreducible fractions are to be accounted zeros.

12. The class $I^2(r,k+1)$ of r-gons L is next to be considered. These have $k+1$ diagonals, two marginal faces, and a sequence of configuration once and but once repeated in the circuit; and they have no axis of reversion, so that the upper face is not identical with the lower. Such a figure L has $x+2$ angles not occupied by diagonals. If x be erased, leaving only those in the marginal triangles, the figure becomes an r'-gon ($r'=r-x$) L′ which has still a sequence once repeated in the circuit, and may or may not have two axes of reversion; for every one of $R^2(r',k+1)$ has such a sequence. If now we pare away the margin of L′, we obtain the $(2k+4)$-gon L″ having k quadrilaterals and two marginal triangles.

13. When k is odd there is a central quadrilateral in L″. Of the $\frac{1}{2}(k-1)$ edges between that and the marginal

triangle let all but y vanish, and in these y quadrilaterals let all the opposite edges vanish except z. If these operations be repeated in the circuit of the r'-gon on the other side of the central quadrilateral, we shall complete

$$\frac{(\frac{1}{2}(k-1))^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{z\,|\,-1}}{1^{z\,|\,1}} \text{ figures,}$$

among which L′ will be twice constructed, *i.e.* with either face uppermost, if the spared edges,

$$\frac{1}{2}(k-1)+z=\frac{1}{2}(r'-6), \text{ or}$$
$$z=\frac{1}{2}(r'-k-5).$$

If now, in each of these

$$\Sigma_y \frac{(\frac{1}{2}(k-1))^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{\frac{1}{2}(r'-k-5)\,|\,-1}}{1^{\frac{1}{2}(r'-k-5)\,|\,1}} \; r'\text{-gons,}$$

we add $\frac{1}{2}x$ points in $\frac{1}{2}(r'-2)$ positions, viz. about the vertex of a marginal triangle, on a side of the central quadrilateral, and on the $\frac{1}{2}(r'-6)$ edges lying between those two positions; and if we repeat the distribution of $\frac{1}{2}x$ points in order round the circuit on the remaining half of the r'-gon, we shall complete upon each r'-gon

$$\frac{(\frac{1}{2}x+1)^{\frac{1}{2}(r'-4)\,|\,1}}{1^{\frac{1}{2}(r'-4)\,|\,1}} = \frac{\frac{1}{2}(r'+x-4)^{\frac{1}{2}(r'-4)\,|\,-1}}{1^{\frac{1}{2}(r'-4)\,|\,1}} \text{ figures,}$$

which will all have a sequence repeated in the circuit. If the r'-gon on which we operate has not two axes of reversion, that is, if $z<\frac{1}{2}(k-1)$ or $r'<2k+4$, all our results will be of the class $I^3(r, k+1)$, for the addition of the x points has added no axis of reversion to the figure; but if the r'-gon should have two axes of reversion, *i.e.* if $r'=2k+4$, the x points, being distributed in every way, may be so arranged as to preserve the symmetry about both axes, while they still exhibit a sequence repeated in the circuit. That is, we shall have constructed both twice every one L′ of the class $I^2(r, k+1)$, which can be reduced by the vanishing of x points to the $(2k+4)$-gon, and also once every one of the class $R^2(r, k+1)$ which has a central face; that is, the product of the last written function of

r' and x into the preceding sum Σ_y, taken for all values of y and of $x=r-r'$, is the number

$$2\mathrm{I}^2(r,k+1)+\mathrm{R}^2(r,k+1),$$

when k is odd.

The sum Σ_y by the reasoning of Art. 4 is seen to be

$$\Sigma_y \frac{\frac{1}{2}(k-1)^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{\frac{1}{2}(r'-k-5)\,|\,-1}}{1^{\frac{1}{2}(r'-k-5)\,|\,1}} = \frac{(\frac{1}{2}(k-1))^{\frac{1}{2}(r'-k-5)\,|\,-1}}{1^{\frac{1}{2}(r'-k-5)\,|\,1}} \cdot 2^{k-\frac{1}{2}r'+2}$$

wherefore, when k is odd, by Art. 6,

$$\mathrm{I}^2(r,k+1)$$

$$=\frac{1}{2}\left\{\Sigma_x \frac{\frac{1}{2}(r-4)^{\frac{1}{2}(r-x-4)\,|\,-1}}{1^{\frac{1}{2}(r-x-4)\,|\,1}} \cdot \frac{(\frac{1}{2}(k-1))^{\frac{1}{2}(r-x-k-5)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-5)\,|\,1}} \cdot 2^{k-\frac{1}{2}(r-x)+2}\right.$$

$$\left. - \frac{\left(\frac{r-2k-2}{4}\right)^{\frac{k+1}{2}\,|\,1}}{1^{\frac{k+1}{2}\,|\,1}} - \frac{\left(\frac{r-2k}{4}\right)^{\frac{k+1}{2}\,|\,1}}{1^{\frac{k+1}{2}\,|\,1}}\right\},$$

where $x \lessdot 0$, $x \lessdot r-2k-4$, $x \gtrdot r-k-5$, and all fractions irreducible are zeros, *i.e.* r and x are even and k is odd.

14. When k is even, there is a central diagonal in L″. Of the $\frac{1}{2}k$ edges of quadrilaterals between this and the marginal triangle let all but y vanish, and of these y quadrilaterals let all the opposite edges but z vanish. Repeating these operations in order in the circuit in the other half of the figure, we complete

$$\frac{(\frac{1}{2}k)^{y\,|\,-1}}{1^{y\,|\,1}} \cdot \frac{y^{z\,|\,-1}}{1^{z\,|\,1}} \text{ figures,}$$

among which L′ is twice made, if the spared edges

$$\frac{1}{2}k+z=\frac{1}{2}(r'-4), \text{ or}$$
$$z=\frac{1}{2}(r'-k-4).$$

In each of these figures we can add $\frac{1}{2}x$ points in $\frac{1}{2}(r'-2)$ positions, viz., about the vertex of a marginal triangle, and in $k=\frac{1}{2}(r'-4)$ consecutive sides of quadrilaterals. We repeat this distribution of $\frac{1}{2}x$ points in the other half of the circuit in order, and thus complete on each r'-gon

$$\frac{(\frac{1}{2}x+1)^{\frac{1}{2}(r'-4)\,|\,1}}{1^{\frac{1}{2}(r'-4)\,|\,1}} = \frac{\frac{1}{2}(r'+x-4)^{\frac{1}{2}(r'-4\,|\,-1}}{1^{\frac{1}{2}(r'-4)\,|\,1}} \text{ figures.}$$

Proceeding exactly as in the preceding article with $k+1$ instead of k (except in the symbols $\mathrm{I}^2(r,k+1)$ and $\mathrm{R}^2(r,k+1)$) we arrive at the result, that when k is even,

$$\mathrm{I}^2(r,k+1)$$
$$=\tfrac{1}{2}\left\{\Sigma_x \frac{\frac{1}{2}(r-4)^{\frac{1}{2}(r-x-4)\,|\,-1}}{1^{\frac{1}{2}(r-x-4)\,|\,1}}\cdot\frac{(\frac{1}{2}k)^{\frac{1}{2}(r-x-k-6)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-6\,|\,1}}\cdot 2^{k-\frac{1}{2}(r-x)+3}\right.$$
$$\left.-\frac{\left(\frac{r-2k}{4}\right)^{\frac{k}{2}|1}}{1^{\frac{k}{2}|1}}\right\},$$

where $x\lessdot 0$, $x\lessdot r-2k-6$, $x\gtrdot r-k-6$; and, as before, irreducible fractions are to be reckoned zero, *i.e.* r, x and k are all even.

15. The enumeration of the classes $\mathrm{R}^2(r,k+1)$, $\mathrm{R}(r,k+1)$ and $\mathrm{I}^2(r,k+1)$ being effected in terms of r and k, we readily obtain that of the asymmetrical and most numerous class $\mathrm{I}(r,k+1)$ of r-gons having two marginal faces and $k+1$ diagonals, by the formula of Art. 5, which gives us

$$\mathrm{I}(r,k+1)=\tfrac{1}{4}\left\{\Sigma_x \frac{k^{r-x-k-4\,|\,-1}}{1^{r-x-k-4\,|\,1}}\cdot\frac{(r-3)^{r-x-3\,|\,-1}}{1^{r-x-3\,|\,1}}\cdot 2^{2k-r+x+4}\right.$$
$$\left.-(\mathrm{R}^2+2\mathrm{R}+2\mathrm{I}^2)(r,k+1)\right\},$$

where $x\lessdot 0$, $x\lessdot r-2k-4$, $x\gtrdot r-k-4$.

This is the number of asymmetrically partitioned r-gons having $k+1$ diagonals and two marginal faces. The numbers denoted by R^2, R and I^2 are given by the formulæ following, which it may be useful to collect into one view.

$$\mathrm{R}^2(r,k+1)$$
$$=\frac{\left(\frac{r-2k-2}{4}\right)^{\frac{k+1}{2}|1}}{1^{\frac{k+1}{2}|1}}+\frac{\left(\frac{r-2k}{4}\right)^{\frac{k+1}{2}|1}}{1^{\frac{k+1}{2}|1}}+\frac{\left(\frac{r-2k}{4}\right)^{\frac{k}{2}|1}}{1^{\frac{k}{2}|1}};\ \text{(Art. 6).}$$

This is the number of r-gons partitioned by $k+1$ diagonals, so as to have two marginal faces and two axes of

reversion or symmetry, one perpendicular to the other. Of these three fractions two or all will be irreducible, and therefore to be accounted zeros.

$$\mathrm{R}^{ag}(r,k+1)$$
$$=\tfrac{1}{2}\Sigma_x\left\{\frac{(\frac{1}{2}r-1)^{\frac{1}{2}(r-x)-1\,|\,-1}}{1^{\frac{1}{2}(r-x)-1\,|\,1}}\cdot\frac{(\frac{1}{2}(k-1))^{\frac{1}{2}(r-x-k-5)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-5)\,|\,1}}\cdot 2^{k-\frac{1}{2}(r-x)+2}\right.$$
$$\left.-\frac{(\frac{1}{4}(r-2k))^{\frac{k+1}{2}\,|\,1}}{1^{\frac{k+1}{2}\,|\,1}}\right\}\ \text{(Art. 8)},$$

where $x\not<0$, $x\not<r-2k-4$, $x\not>r-k-5$. Here k is odd, and r and x both even.

This is the number of r-gons partitioned by $k+1$ diagonals so as to have one agonal axis of reversion, and two marginal faces.

$$\mathrm{R}^{mo}(r,k+1)$$
$$=\Sigma_x\frac{\frac{1}{2}(r-1)^{\frac{1}{2}(r-x-1)\,|\,-1}}{1^{\frac{1}{2}(r-x-1)\,|\,1}}\cdot\frac{(\frac{1}{2}(k-1))^{\frac{1}{2}(r-x-k-4)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-4)\,|\,1}}\cdot 2^{k-\frac{1}{2}(r-x-3)},$$

(Art. 9), where k is odd, r is odd, and x is even, $x\not<0$, $x\not<r-2k-3$, $x\not>r-k-4$.

This is the number of r-gons partitioned by $k+1$ diagonals so as to have two marginal faces and a monogonal axis.

$$\mathrm{R}^{di}(r,k+1)'$$
$$=\tfrac{1}{2}\Sigma_x\left\{\frac{(\frac{1}{2}(r-2))^{\frac{1}{2}(r-x-2)\,|\,-1}}{1^{\frac{1}{2}(r-x-2)\,|\,1}}\cdot\frac{(\frac{1}{2}k)^{\frac{1}{2}(r-x-k-4)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-4)\,|\,1}}\cdot 2^{k-\frac{1}{2}(r-x)+2}\right.$$
$$\left.-\frac{(\frac{1}{4}(r-2k))^{\frac{k}{2}\,|\,1}}{1^{\frac{k}{2}\,|\,1}}\right\}\ \text{(Art. 10)},$$

where r, k and x are all even, $x\not<0$, $x\not<r-2k-4$, $x\not>r-k-4$.

This is the number of $(k+2)$-partitioned r-gons that have a drawn diagonal axis and two marginal faces.

$$\mathrm{R}^{di}(r, k+1)''$$

$$= \tfrac{1}{2}\Sigma_x \left\{ \frac{(\frac{1}{2}(r-2))^{\frac{1}{2}(r-x-2)\,|\,-1}}{1^{\frac{1}{2}(r-x-2)\,|\,1}} \cdot \frac{(\frac{1}{2}(k+1))^{\frac{1}{2}(r-x-k-5)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-5)\,|\,1}} \cdot 2^{k-\frac{1}{2}(r-x)+3} \right.$$

$$\left. - \frac{(\frac{1}{4}(r-2k-2))^{\frac{k+1}{2}\,|\,1}}{1^{\frac{k+1}{2}\,|\,1}} \right\} \text{ (Art. 11),}$$

where k is odd, r and x are even, $x \not> 0$, $x \not> (r-2k-6)$ $x \not< (r-k-5)$.

This is the number of $(k+2)$-partitioned r-gons which have an undrawn diagonal axis and two marginal faces.

$$\mathrm{I}^{2}(r, k+1)(k \text{ odd})$$

$$= \tfrac{1}{2}\Sigma_x \left\{ \frac{(\frac{1}{2}(r-4))^{\frac{1}{2}(r-x-4)\,|\,-1}}{1^{\frac{1}{2}(r-x-4)\,|\,1}} \cdot \frac{(\frac{1}{2}(k-1))^{\frac{1}{2}(r-x-k-5)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-5)\,|\,1}} \cdot 2^{k-\frac{1}{2}(r-x)+3} \right.$$

$$\left. - \frac{(\frac{1}{4}(r-2k-2))^{\frac{k+1}{2}\,|\,1}}{1^{\frac{k+1}{2}\,|\,1}} - \frac{(\frac{1}{4}(r-2k))^{\frac{k+1}{2}\,|\,1}}{1^{\frac{k+1}{2}\,|\,1}} \right\} \text{ (Art. 13),}$$

where r and x are even, $x \not< 0$, $x \not< (r-2k-4)$, $x \not> (r-k-5)$.

This is the number of $(k+2)$-partitioned r-gons doubly irreversible, when k is odd. Their number when k is even, the marginal faces being in either case two, is

$$\mathrm{I}^{2}(r, k+1)(k \text{ even})$$

$$= \tfrac{1}{2}\Sigma_x \left\{ \frac{(\frac{1}{2}(r-4))^{\frac{1}{2}(r-x-4)\,|\,-1}}{1^{\frac{1}{2}(r-x-4)\,|\,1}} \cdot \frac{(\frac{1}{2}k)^{\frac{1}{2}(r-x-k-6)\,|\,-1}}{1^{\frac{1}{2}(r-x-k-6)\,|\,1}} \cdot 2^{k-\frac{1}{2}(r-x)+3} \right.$$

$$\left. - \frac{(\frac{1}{4}(r-2k))^{\frac{k}{2}\,|\,1}}{1^{\frac{k}{2}\,|\,1}} \right\} \text{ (Art. 14),}$$

where r, k and x are even, $x \not< 0$, $x \not< (r-2k-6)$, $x \not> r-k-6$.

In all the above formulæ irreducible fractions are to be counted zeros.

16. The expressions for the $(k+3)$-partitions of the r-gon which have three marginal faces, can be easily obtained in terms of r and k by a process little differing from that above pursued. But I do not see how the re-

quired formulæ are to be found for the $(k+m)$-partitions, which have m marginal faces and $k+m-1$ diagonals. I have however a method of enumerating these partitions for all values of m, which is far better for the purpose of computation than the general expressions in terms of r, k and m would be, if they could be assigned. It is evident, from what is here done in the case of $m=2$, that the sought formulæ will be of very great complexity, unless it can be shown that the series Σ_x which have been above investigated have some simple involution in which x does not appear. When this involution is discovered, it will be worth the while to continue the investigation for higher values of m. Meanwhile, if the k-partitions of the r-gon are required, without classification as to their marginal faces, I know of no method of enumeration more simple than that which I have given in the memoir referred to in the first article, in the *Philosophical Transactions*.

17. The problem of k-partitions of the area of the r-gon, viewed in its completeness, is the problem of *reticulations*, in which the partitioning lines may or may not meet in certain points within the r-gon. These points may be termed *the nodes of the reticulation*. The solution of the problem of the *j-nodal k-reticulations* of the r-gon presents difficulties not less formidable than those which have thus far bid defiance to our analysis in the theory of the polyedra. When $j=0$, we have the question of simple partition by $k-1$ diagonals; when $j>0$, there are j points within the r-gon, in each of which three or more of the partitioning lines meet — some of these lines being diagonals, some passing through a node and an angle of the r-gon, and others through two nodes. The k reticulations are k smaller polygons, all having only angles of less than 180°, into which the r-gon is divided. I have succeeded in the discovery of a general *inductive method* of enumera-

ting the j-nodal k-reticulations of the r-gon, which involves no tentative process, nor any reference to figures. The subject is far too extensive to be here discussed, and the process of computation is very tedious, owing to the great number of formulæ due to the various symmetry and generation of the results, which, for small values of r, j, and k, are collected in but small instalments. I shall content myself here with giving a complete account of the 7-reticulations of the pentagon; and I think it highly probable that any mathematician, who may verify my propositions, will fully satisfy himself as to how far I am in possession of the whole theory.

1. The entire number of 7-reticulations of the pentagon is 7778, of which 413 are symmetrical.
2. Of these, the 4-nodal 7-reticulations are 62 symmetrical and 1010 unsymmetrical.
3. The 5-nodal 7-reticulations are 85 symmetrical and 2000 unsymmetrical.
4. The 6-nodal 7-reticulations are 99 symmetrical and and 2282 unsymmetrical.
5. The 7-nodal 7-reticulations are 69 symmetrical and 1340 unsymmetrical.
6. The remainder have more than 7 nodes, or less than 4.

In this enumeration no figure is counted which is either the repetition or the reflected image of any other.

18. I have to correct two oversights in the sixteenth and twenty-second articles of my Paper "On the triedral partitions of the x-ace," in the present volume.

In the sixteenth article, not only doubly irreversibles but also the doubly reversibles of Art. 15 are constructed, so that these ought to be subtracted before dividing by two.

In the twenty-second article, the triply reversibles of Art. 20, as well as triply irreversibles, are constructed; it

is therefore necessary to substract those triply reversibles, and then to divide by two, for the reason given in the sixteenth article.

The formulæ which have to be corrected in consequence in Art. 25 are (16), (19), (22) and (23). The reader will kindly alter them to the following:

(16). $$\mathrm{I}^2\mathrm{R}^2(4x+m+1,2x)=\mathrm{R}^2(4x+1,2x)$$
$$\times\left\{2^{\frac{m-2}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}-2^{\frac{m-4}{4}}\cdot\frac{\left(\frac{m}{4}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\right\}.$$

(19). $$\mathrm{IR}^2(4x+m+1,2x)=\mathrm{R}^2(4x+1,2x)$$
$$\times\left\{2^{m-2}\cdot\frac{(m+1)^{4x-4-1}}{1^{4x-4\,|\,1}}+2^{\frac{m-4}{4}}\cdot\frac{\left(\frac{m}{4}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\right.$$
$$-2^{\frac{m-2}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}-2^{\frac{m-4}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}$$
$$\left.-2^{\frac{m-2}{2}}\cdot\frac{\left(\frac{m+1}{2}\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\right\}.$$

(22). $$\mathrm{I}^3\mathrm{R}^3(6x+m+1,3x)=\mathrm{R}^3(6x+1,3x)$$
$$\times\left\{\frac{\left(\frac{m}{3}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\cdot 2^{\frac{m-2}{3}}-\frac{\left(\frac{m}{6}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\cdot 2^{\frac{m-6}{6}}\right\}.$$

(23). $$\mathrm{IR}^3(6x+m+1,3x)=\tfrac{1}{3}\mathrm{R}^3(6x+1,3x)$$
$$\times\left\{2^{m-1}\cdot\frac{(m+1)^{6x-4\,|\,1}}{1^{6x-4\,|\,1}}+2^{\frac{m}{3}}\cdot\frac{\left(\frac{m}{6}+1\right)^{x-2\,|\,1}}{1^{x-2\,|\,1}}\right.$$
$$\left.-3\cdot 2^{\frac{m-2}{2}}\cdot\frac{\left(\frac{m}{2}+1\right)^{3x-3\,|\,1}}{1^{3x-3\,|\,1}}-2^{\frac{m-3}{3}}\cdot\frac{\left(\frac{m}{3}+1\right)^{2x-2\,|\,1}}{1^{2x-2\,|\,1}}\right\}.$$

The formulæ of Art. 25 thus corrected, together with those of Art. 26, are all that are required for the enumeration of the triangular partitions of the r-gon for all values of r.

XVI. — *A Contribution to Urology, embracing Observations on the Diurnal Variations in the Acidity of the Urine, chiefly in relation to Food.*

By William Roberts, B.A., M.D.,
Physician to the Manchester Royal Infirmary.

Read May 3rd, 1859.

It has been the universal belief, until recently, that the reaction of the human urine was, in health, invariably acid; and that a neutral or alkaline condition of it was either a sign of disease, or the consequence of partaking of alkaline substances, or of subacid fruits, the vegetable salts of which, being broken up in the blood, appeared in the urine as alkaline carbonates.

In 1845 Dr. Bence Jones* called this belief in question, and announced that he found that the urine became not unfrequently alkaline a few hours after taking food.

Three years later Dr. Jones investigated the subject comprehensively; and the result of his inquiries were published in the *Philosophical Transactions* for 1849. He conducted the inquiry as follows: — The urine was passed immediately before breakfast (about nine), and the acidity per 1000 grains estimated by a test solution of carbonate of soda. After breakfast the urine was passed about every hour or hour-and-half, and the products separately examined by the test solution. This went on till dinner (about six p.m). After dinner the observations were less frequent and regular. Generally the first examination was

* *Phil. Trans.* 1845, p. 349.

made about two hours after the meal, and the next about three hours later. The urine of sleep and of the following morning were also separately examined. As a rule, about twelve distinct observations were made in each period of twenty-four hours. The effect of ordinary mixed food, of purely animal and of purely vegetable diet, was investigated during separate days on which one or other class of diet was exclusively used. He sums up his results in the following propositions: —

I. "As regards the variations of the acidity of the urine for three days on mixed diet. The acidity soon after food was found to decrease and to attain its lowest limit from three to five hours after breakfast and dinner; sooner however after breakfast than after dinner. The acidity then gradually increased and attained its highest limit just before food. If no food was taken the acidity of the urine did not decrease, but remained nearly the same for twelve hours. It fell immediately after food was taken.

II. "When animal food only was taken, the diminution of the acidity after food was more marked and more lasting than when a mixed diet was taken; and the acidity before food rose rather higher with a mixed diet than it did with animal food.

III. "When vegetable food only was taken the contrast with animal food was very marked. The urine did not decrease in acidity to the same degree; though it became neutral it did not become highly alkaline. The increase in the acidity of the urine was by no means so marked as the decrease of the alkalescence. The acidity of the urine was rather higher with the vegetable food than it was with animal food.

IV. "The result of these experiments is, that the acidity of the urine is always changing, and that the changes depend on the state of the stomach. When much acid is in the stomach the acidity is then diminished. As the acid

returns from the stomach the acidity of the urine increases, and generally reaches its highest limit before food is again taken."*

These views have not obtained universal assent. It has been objected to them that they are based on observations on a single individual,† and cannot therefore be safely applied generally. And not only have no confirmatory facts been advanced, but subsequent observations have tended to re-establish the old opinion, that a neutral or alkaline state of the urine is an abnormal one. Dr. Beneke‡ examined his own urine on twenty-three days, and failed to detect any depression of the acidity as a result of taking food. In addition, he made not less than a hundred day-observations on different sick and healthy persons with this result — that, although the urine did occasionally show a depressed acidity and even an alkaline state after a meal, and especially after breakfast, yet this was very far from being constantly the case.

Dr. Julius Vogel§ rejects Dr. Jones's experiments on

* *Phil. Trans.*, vol. lvi. p. 244.

† Although the experiments detailed in Dr. Jones's Paper were performed on a single individual, additional evidence of the effect of food on the reaction of the urine was advanced by him in the *Philosophical Transactions* for 1845, p. 345. He there says: "Dr. Andrews of Belfast stated to me, that having observed a case otherwise in perfect health, in which the urine was almost invariably alkaline about two hours after breakfast, so much so as frequently to be loaded with a deposition of phosphates whilst still in the bladder, he was led to observe the urine of about fifteen students in good health immediately after it was voided about noon. He found it to be alkaline in about two-thirds of the cases." "At the present time I know five physicians in whom the above phenomena at this period of the day are more or less frequently visible in a greater or lesser degree, and in London this alkalescence will be found in those who are considered generally healthy much oftener than is imagined."

‡ *Archiv des Vereins für gem. Arb. zur Förd. der Wissensch. Heilkunde*, vol. i. p. 438.

§ Neubauer und Vogel, *Anleitung zur Analyse des Harns*. 2nd edition, p. 175.

the ground that all his determinations were reckoned for 1000 parts instead of per hour. This objection would have had weight if the density of each specimen had not been recorded. Great dilution of the urine from abundant potation would no doubt reduce the *degree* of acidity per 1000 parts, even when the quantity of acid discharged per hour remained constant; but this source of fallacy was guarded against in the observations of Dr. Jones by taking the density of the secretion — this being a sufficient measure of its concentration. Dr. Vogel goes on to say: "Researches undertaken partly by myself, and partly by others under my direction, showed uniformly that the greatest quantity of acid secreted per hour by the kidneys occurred during the night, the least in the forenoon, while a medium quantity was discharged in the afternoon after the principal meal. These results therefore are unfavourable to the conclusions of Dr. Bence Jones, but do not tell conclusively against them, inasmuch as other circumstances may have had an influence on the amount of acidity."

Dr. Sellers, in the *Edinburgh Medical Journal* for January 1859, states that in a good many trials he has not been able to satisfy himself "that the rule, as laid down by Dr. B. Jones, is generally applicable in Edinburgh; certainly not to the extent that the urine loses entirely its acid character, or that it becomes alkaline." Nevertheless it has seemed to him "that the variations in the degree of its acidity are in some measure governed by the existing states of the stomach."

Dr. Delavaud (*Gaz. Médicale*, 1851, No. 44) found the urine becoming neutral or alkaline after breakfast, but not after dinner.

Seeing this discrepancy in the results obtained by different observers, it seemed not undesirable to seek additional and exact information on the effect of food on the reaction

of the urine, and to find out some means, if it were possible, of reconciling the conflicting facts. The following experiments were undertaken with that purpose; and in a subsequent portion of the Paper some considerations are advanced which go far to account for this want of agreement.

All the experiments herein detailed concerned a single individual. He was a healthy man, twenty-eight years of age, taking moderate exercise, living in most favourable hygienic conditions, and weighing 144 lbs.

In order to ascertain the exact amount of change undergone by the urine after food, it was thought essential to collect the secretion at each hourly period succeeding a meal, and by measuring its quantity and saturating power to obtain data from which the precise amount of free acid or free alkali separated per hour by the kidneys could be estimated. At periods more remote from meal-times the urine was usually collected every two hours. As compared with Dr. Jones's method of merely ascertaining the degree of acidity or alkalescence per 1000 grains, it had this important advantage, that it eliminated the inaccuracies consequent on the great inequality of concentration to which the urine is subject from food, drink, exercise and sleep, whereby its quantity and aqueousness rise or fall immensely, and with great suddenness.

The following particulars were taken of each urine, and arranged in a tabular form: —

I. *The Time of Day during which it was secreted.*

II. *The Quantity.* This was estimated in a glass vessel graduated on the scale of the 1000-grain measure. If the period of secretion exceeded or fell short of an hour by five, ten, or fifteen minutes, or if the period was two or more hours, as during sleep, the hourly rate of secretion was exactly calculated from the quantity and the interval. The results are arranged in the second columns of the

tables, and show in grain-measures the hourly rate of flow of the urine. From this volume-measure and the density the *weight* of urine per hour can be readily calculated.

III. *The Density.* A gravimeter of tried accuracy was usually employed. When the quantity was too scanty for the instrument, a 250-grain specific gravity bottle was substituted; and in one set of experiments of seven days (Table III.) the bottle was exclusively used.

IV. *The Solid Matters.* The amount of solid residue per 1000 grains was calculated according to Christison's formula.* A second calculation from this and the hourly quantity gave, in grains, the solids separated per hour. These are arranged in the fourth columns. It is not pretended that these figures represent with accuracy the actual amounts, but as relative values they may be fairly assumed as near the truth. The results obtained are exceedingly uniform, considering the somewhat complicated, and avowedly uncertain, calculation on which they are based; and the remarkable conclusion to which they point is indicated with great distinctness.

V. *The Reaction.* The degree of acidity or alkalinity was ascertained by a test solution after the usual method in volumetrical analyses. For the former a solution of caustic soda was employed, and for the latter dilute sulphuric acid. The two solutions were made of equal saturating power; each 100 grain measures being equivalent to 1 grain of dried carbonate of soda. The results are arranged in double columns, showing separately, in grains of dried carbonate of soda, the degree of acidity or alkalescence per 1000 grain-measures, and the amounts per hour. 500 grains of urine were usually operated upon; but if the urine was very dilute 1000 grains; and if very concentrated and scanty 250 grains were employed.

VI. *The Appearance of the Urine.* The condition of the

* See table in Bird's *Urinary Deposits*, 5th edition, p. 60.

urine as to clearness or turbidity, *when passed*, and the colour, were also recorded. Frequently, too, the appearance on cooling or standing was noted, but the results do not appear in the following tables.

To complete the record, the times of the meals and their nature were chronicled.

The condition of the body as to exercise, occupation and sleep was also notified, but inasmuch as a single statement will suffice for all the particulars, the details are not here recorded, so as not too greatly to complicate the tables.

The mode of life of the subject of experiment was kept as nearly as possible uniform during the time of observation. He usually rose at seven, breakfasted at eight, dined at two, sometimes at four; and took no further solid food until breakfast next morning. He retired to rest at one in the morning; so that when the days of observation were successive there were but six hours of sleep. As to occupation and exercise there were necessarily some variations, but these were reduced to a minimum. Variously engaged in-doors until ten or eleven in the morning, moderate out-of-door exercise was afterwards taken until one or two. After dinner occupation was sedentary for two or three hours; then moderate out-of-door exercise was taken for one or more hours. Care was taken to avoid any violent or protracted exertion on the one hand or a complete inactivity on the other. Walking exercise for an hour or two did *not* produce any perceptible effect on the results.

The experiments are divided into six sets; each set embracing from three to seven days; and the results for the separate days are collected into a single table of averages, which shows the mean quantities for the days composing the set. The observations on the reaction, however, are given in detail for each separate day, and form a companion table to each table of averages.

The first observations were made on ordinary mixed food. They form two sets of four and seven days respectively, and there are added some odd days, which could not be united together to form a distinct set.

Effect of Mixed Diet.

The first set embraces four days, on which hourly observations were made from seven a.m. to eight a.m. next morning (except of course the hours of sleep, which form a single observation), a period of twenty-five hours.

TABLE I. Mixed food. Breakfast at Eight, dinner at Four. Mean of four days, not consecutive.
(viz. Sept. 30, Oct. 4, Oct. 13, Oct. 21, 1858.)

Time of day	Hourly flow In grains	Density	Solids per hour In grains	Acidity Per 1000 grain measure In grs. carb.	Acidity Per hour of dried soda	Alkalinity Per 1000 grain measure In grs. carb.	Alkalinity Per hour of dried soda	Appearance	Remarks	Diet
7–8	514	1023·4	27·70	1·70	0·83	...	...	Each time clear, rich amber.		Breakfast at 8. Tea with dried toast and broiled bacon, or eggs, or pork chop.
8–9	770	1023·5	40·61	0·92	0·66	...	...	Each time clear, amber.		
9–10	902	1021·0	43·38	...	...	0·64	0·60	Twice clear and twice cloudy, pale greenish amber.	Three times alkaline and once neutral.	
10–11	1025	1019·8	46·57	0·22	0·22	0·87	0·94	Three times clear, once cloudy, pale greenish amber.	Twice alkaline, twice faintly acid.	
11–12	1125	1017·6	42·39	0·39	0·33	0·38	0·55	Each time clear, paler amber.	Three times acid, once faintly alkaline.	
12–1	915	1020·0	40·78	0·45	0·35	...	...	Clear, pale amber.		
1–2	1087	1017·0	36·38	0·66	0·62	...	...	Clear, pale amber.		
2–3	772	1020·8	32·71	1·00	0·64	...	...	Clear, pale amber.		
3–4	882	1020·4	29·87	1·24	0·77	...	...	Clear, pale amber.		
4–5	475	1024·3	26·39	1·19	0·58	...	...	Clear, pale amber.		Dinner at 4. Meat, potatoes, bread, cheese, beer.
5–6	2605	1006·2	31·97	0·20	0·40	0·58	1·37	Clear, pale straw.	Three times acid and once strongly alkaline.	
6–7	960	1016·5	34·08	...	...	0·99	1·00	Muddy, yellowish.		
7–8	912	1018·7	37·28	...	...	0·83	0·87	Very muddy, yellowish.		
8–9	1095	1019·0	46·08	...	...	0·53	0·60	Muddy, yellowish.		At 9 two cups of coffee, without sugar or cream.
9–10	1740	1014·4	44·95	0·40	0·49	0·26	0·58	Clear, pale amber.	Once neutral, once alkaline, twice acid.	
10–11	1505	1014·6	43·01	0·82	1·06	...	...	Clear, pale amber.	(Two days).	
11–12	605	1021·2	29·78	1·88	1·11	...	...	Clear, pale amber.	(Two days).	
12–1	510	1023·4	28·52	1·77	0·89	...	...	Clear, pale amber.	(Two days).	
1–7	681	1015·0	19·73	1·30	0·74	...	...	Clear, reddish amber.	(Asleep).	
7–8	355	1022·0	17·93	1·74	0·59	...	...	Clear, rich amber.	(Two days).	

On these four days the urine became alkaline both after breakfast and after dinner; and the effect of dinner is observed to be more intense, as well as more enduring, than that of breakfast. In constructing this and the succeeding tables of means a difficulty arose as to how to deal with those hours which on some of the days exhibited an acid, and on others an alkaline urine, such for example as from ten to eleven and eleven to twelve. The plan adopted has been to take a separate mean for the acid days and a separate mean for the alkaline days. This will explain why at these hours the urine is made to appear both acid and alkaline. Any obscurity or inconvenience arising from this will be obviated by a study of the companion tables, which give *in extenso* the daily results for the acidity and alkalinity.

TABLE II. exhibits the hourly Variation of Reaction for the days composing Table I. The Plus sign is prefixed when the urine was Acid, and the Minus sign when Alkaline.

	Sept. 30, First day		Oct. 4, Second day		Oct. 13, Third day		Oct. 21, Fourth day		Diet
	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	
7–8	+1·88	+0·90	+2·16	+0·81	+1·20	+0·83	+1·56	+0·80	Breakfast at 8.
8–9	+0·90	+0·77	+1·46	+0·94	+0·28	+0·21	+1·00	+0·74	Tea with sugar and cream; dried toast with broiled bacon, or eggs, or pork chop.
9–10	−1·00	−0·95	−0·64	−0·55	−0·84	−0·81	0·00	0·00	
10–11	−1·32	−1·45	+0·20	+0·21	−0·42	−0·44	+0·24	+0·22	
11–12	−0·38	−0·55	+0·68	+0·45	+0·09	+0·13	+0·42	+0·42	
12–1	0·00	0·00	+0·94	+0·52	+0·16	+0·15	+0·72	+0·75	
1–2	+0·41	+0·28	+1·18	+0·62	+0·64	+0·81	+0·42	+0·78	
2–3	+0·22	+0·15	+2·28	+1·03	+0·88	+0·68	+0·62	+0·72	
3–4	+0·32	+0·30	+2·52	+0·76	+1·40	+0·77	+0·72	+1·25	
4–5	+1·28	+0·66	+2·36	+0·83	+0·80	+0·40	+1·56	+0·83	Dinner at 4. Meat, potatoes, bread, cheese, beer. At 9 two cups of coffee without sugar or cream.
5–6	+0·06	+0·28	+0·26	+0·33	−0·58	−1·37	+0·28	+0·59	
6–7	−0·84	−0·88	−0·20	−0·15	−1·48	−1·30	−1·46	−1·68	
7–8	−1·30	−1·79	−0·76	−0·58	−1·00	−0·97	−0·25	−0·14	
8–9	−0·62	−0·86	−0·78	−0·78	−0·56	−0·58	−0·18	−0·15	
9–10	+0·36	+1·02	+0·86	+0·45	−0·26	−0·58	0·00	0·00	
10–11	+0·60	+1·77	+1·22	+1·05	+1·20	+1·27	+0·44	+0·86	
11–12	+0·54	+2·02	+1·00	+1·55	+2·28	+1·27	+1·48	+0·96	
12–1	+0·96	+1·39			+1·62	+0·89	+1·92	+0·92	
1–7	+1·16	+0·97	+1·70	+1·18	+0·74	+0·69	+1·86	+0·80	
7–8					+1·24	+0·51	+2·24	+0·67	

On the first day a slight luncheon, consisting of two thin pieces of bread and butter and a glass of water was taken at two. This accounts for the slow degrees by which the acidity recovered its ordinary level after breakfast.

About a pint of porter was taken between nine and eleven on the first night, and some bread and butter with tea between nine and ten on the second night. For this reason the urines passed after ten o'clock, on the first and second nights, are not reckoned in the table of means. Previous to these four days the condition of the urine had been examined at short intervals on five other days after breakfast, and on four days after dinner. On each day the urine became alkaline after dinner. After breakfast it became alkaline three times; but remained acid, in a diminished degree however, on the remaining two days.

The times of change from acid to alkaline, and back again from alkaline to acid, having now been ascertained with tolerable certainty, it was thought unnecessary to carry out so rigidly the very laborious plan of hourly observation throughout the entire day. In the succeeding experiments, therefore, hourly observations were only made at the critical periods, when the reaction was oscillating; but when the acidity was steadily rising, or had attained its usual level, observations were made every two hours. By this modification the experiments could be carried on with comparative ease, and continued for several consecutive days.

The effect of mixed food was again subjected to observation during seven days, all consecutive but one. Two meals a day were taken, and no alcoholic drinks.

TABLE III. Mixed food. Breakfast at Eight, dinner at Two. Mean of seven days, all consecutive but one.
(*March* 8, 9, 10, 11, 12, 14, 15).

Time of day	Hourly flow	Density	Solids per hour	Acidity Per 1000	Acidity Per hour	Alkalinity Per 1000	Alkalinity Per hour	Appearance	Remarks	Diet
7–8	340	1024·91	19·55	1·54	0·52	...	...	Clear, amber; nearly always depositing lithates on cooling or standing.		Breakfast at 8. Meat, coffee or tea, bread and butter.
8–9	520	1024·70	29·27	0·92	0·47	...	...	Clear, amber; generally depositing lithates.		
9–10	850	1020·84	39·22	0·58	0·43	0·95	0·83	Four times cloudy, three clear, yellowish.	Five times alkaline, twice acid.	
10–11	1010	1019·93	44·34	0·34	0·33	0·62	0·66	Yellowish; twice turbid, five times clear, and once depositing lithates.	Three times alkaline, three times acid, and once neutral.	
11–12	1340	1017·58	45·24	0·52	0·47	...	...	Clear, pale amber; only depositing lithates once.	Five times acid, twice neutral.	
12–2	1090	1018·97	41·43	0·87	0·75	...	...	Clear, amber; once depositing lithic acid, and once lithates.		
2–3	730	1024·00	38·69	1·12	0·72	...	...	Clear, amber; twice depositing lithates on cooling.		Dinner at 2. Meat, potatoes, bread, cheese, water.
3–4	3440	1006·66	38·79	0·26	0·45	0·12	0·49	Clear, pale straw.	Three times acid, once neutral, and twice alkaline.	
4–5	1170	1015·62	41·21	0·30	0·38	1·39	1·45	Six times turbid, once clear, yellowish.	Six times alkaline, once acid.	
5–6	790	1022·89	41·09	0·46	0·32	1·89	1·47	Six times very turbid and yellowish, once clear.	Six times alkaline, once acid.	
6–7	890	1023·27	49·01	0·00	0·00	1·54	1·38	Six times turbid, once clear.	Six times alkaline, once neutral.	
7–9	910	1022·77	47·44	0·55	0·48	...	...	Clear, amber.		On two evenings three cups of water; another evening one cup, and three evenings nothing, either solid or liquid, after dinner.
9–11	850	1021·46	37·66	0·93	0·77	...	...	Clear, amber.		
11–1	610	1022·05	28·53	1·07	0·62	...	...	Clear, amber.		
1–7	350	1023·05	15·53	1·30	0·38	...	...	Clear, reddish amber.	Asleep.	
7–8	320	1024·50	17·75	1·55	0·45	...	...	Clear, rich amber.	(Five days).	

TABLE IV. exhibits the Variations in Reaction for the several days composing Table III.
(The Plus and Minus signs used as before).

Time of day	March 8, First day		March 9, Second day		March 10, Third day		March 11, Fourth day		March 12, Fifth day		March 14, Sixth day		March 15, Seventh day		Diet
	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	
7–8	+1·92	+0·65	+1·12	+0·43	+0·92	+0·35	+1·52	+0·44	+2·00	+0·58	+1·76	+0·65	?	?	
8–9	+1·74	+0·80	+0·22	+0·17	+0·10	+0·05	+1·00	+0·44	+1·04	+0·50	+1·40	+0·78	+0·95	+0·43	Breakfast at 8. Tea, dried toast with meat.
9–10	+0·40	+0·29	−0·82	−1·07	−1·00	−0·88	−0·70	−0·46	−1·12	−0·84	+0·76	+0·58	−1·12	−0·94	
10–11	+0·48	+0·46	−0·42	−0·72	−0·94	−0·89	0·00	0·00	+0·24	+0·20	+0·62	+0·65	−0·52	−0·38	
11–12	+0·42	+0·52	0·00	0·00	0·00	0·00	+0·82	+0·79	+1·00	+0·84	+0·85	+0·76	+0·52	+0·42	
12–2	+0·73	+0·82	+0·15	+0·29	+0·46	+0·83	+1·08	+0·88	+1·40	+0·91	+1·40	+0·80	+0·88	+0·62	
2–3	+1·04	+0·94	+0·41	+0·36	+0·48	+0·46	+1·62	+0·89	+2·04	+1·00	} +0·28	+0·71	+1·16	+0·66	Dinner at 2. Meat, potatoes, bread, cheese, water.
3–4	+0·10	+0·42	−0·15	−0·41	−0·10	−0·48	+0·20	+0·70	+0·76	+0·68	}		0·00	0·00	
4–5	−1·68	−1·28	−2·66	−2·21	−1·16	−2·06	−0·64	−0·80	−0·60	−0·78	+0·30	+0·38	−1·64	−1·74	
5–6	−1·66	−0·83	−4·12	−2·60	−1·60	−1·50	−1·24	−1·20	−0·72	−0·52	+0·46	+0·32	−2·00	−2·14	
6–7	−1·36	−0·98	−2·60	−2·34	−2·08	−1·81	−0·88	−0·88	−1·14	−1·14	0·00	0·00	−1·20	−1·16	
7–9	+0·58	+0·54	+0·12	+0·14	+0·30	+0·37	+0·70	+0·60	+0·84	+0·59	+0·56	+0·46	+0·90	+0·67	
9–11	+0·77	+0·83	+0·40	+0·49	+0·68	+0·88	+1·18	+0·90	+0·92	+0·92	+1·14	+0·72	+1·44	+0·65	
11–1	+0·53	+0·62	+0·21	+0·53	+1·08	+0·65	+1·48	+0·70	+1·56	+0·64	+1·56	+0·59	?	?	
1–7	+0·42	+0·37	+0·85	+0·34	+1·30	+0·47	+1·68	+0·42	+1·48	+0·38	+1·52	+0·29	+1·88	+0·37	
7–8	+1·12	+0·43	+0·92	+0·35	+1·52	+0.44	+2·00	+0·58	?	?	?	?	+1·68	+0·44	

The results obtained on these seven days correspond closely with the foregoing, but they are exhibited in a less exaggerated degree. On the first and sixth days the urine after breakfast maintained its acidity, though on the first it was greatly reduced. On the sixth day the effect of both breakfast and dinner was comparatively small. On all the other days the urine became alkaline after both meals. The succession of events after each meal was quite uniform throughout. In an hour or two the acidity of the urine began to decline, and it sank to a minimum or changed to alkalinity at the second, third, or fourth hours; then, beginning to recover, the acidity gradually increased in degree until it attained its ordinary level. No departure from this sequence of events occurred, even as an exception.

What may be the circumstances which cause the very considerable inequalities between the several days, and between this set of observations and the preceding, — why a meal one day should only produce a slight depression of the acidity, another day render the urine neutral or faintly alkaline, and a third day change it for several hours to a strongly alkaline reaction, — is not capable of complete answer. But several of the disturbing causes have revealed themselves in the course of these observations, and will be discussed later on.

The effect of mixed food was further ascertained on four additional days, on which the times of emission of the urine were so irregular that the day tables could not be collected into one table of averages. On three of these, instead of being confined to two meals a day, the subject of experiment partook of four meals, which was according to his customary habit of life; and at dinner and supper he was allowed a pint of ale.

TABLE V. Mixed food. Four meals a day. A pint of ale with dinner and supper.
(*March* 18).

Time of day	Hourly flow	Density	Solids per hour	Acidity		Alkalinity		Appearance	Remarks	Diet
				Per 1000	Per hour	Per 1000	Per hour			
7·40–9	260	1034·16	20·67	2·40	0·62	...	...	Clear, rich amber.	Depositing lithates abundantly on cooling.	Breakfast at 8-20. Tea, dried toast, meat.
9–10	550	1027·52	35·20	0·72	0·39	...	...	Clear, rich amber.	Depositing lithates abundantly on cooling.	
10–11	540	1025·24	31·64	0·94	0·51	...	...	Clear, rich amber.	Depositing lithates on standing some hours.	
11–2	460	1025·92	27·78	1·42	0·65	...	...	Clear, rich amber.	Depositing lithates on long standing.	
2–4·30	890	1013·48	27·76	0·72	0·64	...	...	Clear, rich amber.	Not depositing on standing.	Dinner at 2. Meat, potatoes, bread, cheese, ale.
4·30–6	400	1027·50	25·60	...	...	0·30	0·12	Clear.	Not depositing on standing.	
6–7	590	1027·80	38·05	...	...	1·40	0·82	Turbid.	Thick on passing.	Two cups of coffee and three pieces of bread and butter.
7–11	1150	1018·20	48·76	0·38	0·44	...	...	Clear, amber.	Not depositing.	
11–1	1260	1010·56	30·74	0·66	0·38	...	...	Clear, amber.	Not depositing.	At 11-30 bread and cheese and ale.
1–7	830	1009·64	18·92	0·80	0·66	...	...	Clear, amber.	Not depositing.	

TABLE VI. Mixed food. Four meals. A pint of ale with dinner and supper.
(*March* 19).

Time of day	Hourly flow	Density	Solids per hour	Acidity Per 1000	Acidity Per hour	Alkalinity Per 1000	Alkalinity Per hour	Appearance	Remarks	Diet
7–9	330	1026·64	20·39	?	?	...	...	Clear, amber.	Depositing on cooling.	Breakfast at 8. Coffee, dried toast, meat.
9–10	760	1026·10	46·13	0·00	0·00	0·00	0·00	Clear, amber.	Not depositing.	
10–11	970	1021·50	48·50	0·15	0·14	...	...	Clear, amber.	Not depositing.	
11–2	790	1022·50	41·31	0·33	0·26	...	...	Clear, amber.	Not depositing.	
2–4	1430	1010·50	34·74	0·47	0·67	...	...	Clear, pale amber.	Not depositing.	Dinner at 2-15. Meat, potatoes, bread, cheese, ale.
4–6	1140	1017·00	45·14	...	...	0·58	0 66	Clear, yellowish amber.	Not depositing.	
6–8·30	710	1028·00	46·30	...	...	0·32	0·23	Clear, pale amber.	Not depositing.	
8·30–10	1000	1022·50	52·20	0·20	0·20	...	...	Clear, pale amber.	Not depositing.	Tea at 9. Bread and butter.
10–12	1120	1016·50	42·78	0·63	0·71	...	...	Clear, pale amber.	Not depositing.	Supper at midnight. Meat, bread, ale.
12–8	350	1027·00	22·05	1·52	0·53	...	...	Clear, deep reddish amber.	Depositing on cooling.	

TABLE VII. Mixed food. Four meals. A pint of ale with dinner and supper.
(*March* 20).

Time of day	Hourly flow	Density	Solids per hour	Acidity Per 1000	Acidity Per hour	Alkalinity Per 1000	Alkalinity Per hour	Appearance	Remarks	Diet
8–9·30	420	1027·00	26·41	1·28	0·54	...	...	Clear, amber.	Depositing on cooling.	Breakfast at 9. Coffee, meat, bread and butter.
9·30–10·25	750	1025·50	45·47	0·88	0·66	...	...	Clear, amber.	Depositing on cooling.	
10·25–12·25	1230	1020·00	57·31	...	...	0·51	0·63	Clear, amber.	Not depositing.	
12·25–2	1030	?	?	0·14	0·14	...	...	Clear, amber.	Not depositing.	
2–3	4150	1004·00	38·67	0·20	0·83	...	...	Clear, very pale straw.	Not depositing.	Dinner at 1. Meat, bread, cheese, potatoes, greens, ale.
3–5·30	910	1021·50	45·50	...	...	0·58	0·53	Clear, amber.	Not depositing.	
5·30–6·15	950	1023·50	51·96	0·72	0·68	...	...	Clear, amber.	Not depositing.	Two cups of coffee and bread and butter at 5-45.
6·15–8·35	2100	1012·75	59·85	0·43	0·90	...	...	Clear, pale amber.		
8·35–11	2680	1006·50	40·47	0·34	0·90	...	...	Clear, straw.		Supper at 9. Meat, bread, cheese, ale.
11–7	425	1022·50	22·57	1·28	0·54	...	...	Clear, reddish amber.		

TABLE VIII. Mixed food. Two meals a day. Breakfast at 8-45, *dinner at Two. No alcoholic drinks.* (*Jan.* 31).

Time of day	Hourly flow	Density	Solids per hour	Acidity Per 1000	Acidity Per hour	Alkalinity Per 1000	Alkalinity Per hour	Appearance	Remarks	Diet
7–8	330	?	?	2·24	0·74	...	...	Clear, rich amber.	Depositing on standing.	Breakfast at 8-45. Tea, meat, bread and butter.
8–9	400	1030·0	27·96	2·00	0·80	...	...	Clear, amber.	Depositing on standing.	
9–10	670	1027·0	42·71	1·00	0·67	...	...	Clear, amber.	Depositing on standing.	
10–11	700	1025·0	40·74	0·76	0·53	...	...	Clear, pale amber.		
11–12	850	1024·5	48·45	1·04	0·88	...	...	Clear, pale amber.		
12–2	660	1026·0	40·26	2·20	1·45	...	...	Clear, amber.		
2–3	530	?	?	2·33	1·23	...	...	Clear, amber.	Depositing in two hours.	Dinner at 2. Meat, bread, potatoes, cheese, water.
3–4	850	1020·0	39·61	1·12	0·95	...	...	Clear, amber.		
4–5	760	1023·5	41·57	0·00	0·00	0·00	0·00	Faintly cloudy.		
5–6	660	1027·0	41·51	0·00	0·00	0·00	0·00	Faintly cloudy.		
6–7	520	?	?	1·37	0·72	...	...	Clear, amber.		At 9 a cup of weak coffee without sugar or cream.
7–9	1400	1017·0	50·40	1·20	1·68	...	...	Clear, pale amber.		
9–11	570	1029·0	37·47	1·82	1·04	...	...	Clear, amber.		
11–1	3720	1005·0	42·92	0·32	1·19	...	...	Clear, straw.		At 11 two cups of weak coffee without sugar or cream.
1–7	394	1020·0	18·17	1·46	0·56	...	...	Clear, deep reddish amber.		
7–8	375	?	?	1·65	0·62	...	...	Clear, rich amber.		

It is seen by Tables V., VI. and VII. that moderate use of malt liquor did not perceptibly affect the results. A supper of meat and bread with a pint of ale had not the power of lowering the acidity of the night urine; on the contrary, the discharge of acid per hour was considerably above the average in Table III. for the nights on which no supper was taken; and the acidity per 1000 was at least as high as the general average. The effect of breakfast was decidedly less when a meat supper had been taken the night before.

In Table VIII. may be recognized a very unsusceptible day — more so than any met with during the whole course of the observations.

2. *The Effects of purely Vegetable Food.*

Two sets of observations were made with a view of ascertaining how far a diet exclusively composed of vegetable matters (excluding sweet and subacid fruits) affected the reaction of the urine.

In the *First Set,* which included four days, there was always taken a hearty supper of mixed food the night before. Only two of the days were consecutive, the others alternated with days on which a purely animal or a mixed diet was used. The observations extended from seven in the morning until about eleven at night.

TABLE IX. Vegetable food only. Mean of four days, not consecutive. Breakfast at Eight, dinner at Four.

Time of day	Hourly flow	Density	Solids per hour	Acidity		Alkalinity		Appearance	Remarks	Diet
				Per 1000	Per hour	Per 1000	Per hour			
7–8	490	1027·10	31·15	2·08	1·20	...	...	Clear, amber.		
8–9	750	1026·87	47·28	1·80	1·35	...	...	Clear, amber.		Breakfast at 8. Tea or coffee with sugar only, and bread.
9–10	940	1025·00	54·54	1·20	1·14	...	...	Clear, amber.		
10–11	1600	1017·10	58·75	0·63	1·00	...	...	Clear, pale amber.		
11–12	2250	1013·00	56·07	0·56	1·19	...	...	Clear, pale amber.		
12–2	1080	1019·25	50·62	1·23	1·32	...	...	Clear, amber.		
2–4	580	1026·00	34·89	2·30	1·30	...	...	Clear, rich amber.		
4–5	420	1028·00	29·01	3·11	1·29	...	...	Clear, rich amber.		Dinner at 4. Vegetable soup, bread, rice, sugar, celery, water.
5–6	3570	1007·10	36·85	0·49	1·12	...	...	Clear, straw.		
6–7	880	1018 00	36·70	0·90	0·79	...	...	Clear, amber.		
7–8	650	1024·75	37·12	0·85	0·57	...	...	Clear, amber.		
8–9	660	1025·50	38·74	2·39	1·44	...	...	Clear, amber.		
9–11	730	1024·62	40·45	2·19	1·45	...	...	Clear, amber.		

TABLE X. exhibits the Variations in Reaction on the several days composing Table IX.

Time of day	Jan. 20. First day		Jan. 21, Second day		Jan. 25, Third day		Jan. 27, Fourth day		Diet
	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	
7–8	2·10	?	2·90	1·24	2·20	1·08	2·16	1·30	
8–9	1·96	1·76	1·70	1·09	1·92	1·25	1·62	1·33	Breakfast at 8. Tea or coffee, with sugar but no cream, with four or five slices of bread.
9–10	1·36	1·49	1·20	1·02	1·12	0·98	1·14	1·08	
10–11	0·68	1·09	0·80	1·24	0·48	0·93	0·58	0·76	
11–12	0·66	1·32	0·68	1·33	0·34	1·14	0·58	0·99	
12–1	1·32	1·62	1·36	1·15	1·14	1·30	1·12	1·22	
1–2	2·08	1·56							
2–3			2·12	1·07	2·10	1·20	2·80	1·37	
3–4	?	?							
4–5			2·77	1·15	3·78	1·40	2·80	1·34	Dinner at 4. A vegetable soup, bread, rice, sugar, celery, water.
5–6	3·30	1·44	0·20	0·98	1·00	1·11	0·27	1·27	
6–7			0·96	0·71	0·65	0·60	1·10	1·07	
7–8			1·98	1·22	0·42	0·25	1·28	0·89	
8–9	3·70	1·70			1·30	1·24	2·60	1·61	
9–10	3·12	0·85	2·12	1·60	1·28	1·23	2·26	2·15	
10–11					?	?	?	?	

On none of these four days did the urine become alkaline either after breakfast or dinner; indeed the hourly discharge of acid suffered scarcely an appreciable declension after breakfast on the first two days. And throughout the entire set, the effect of the meals was strikingly less than with mixed food. Small as the effect was however, its reality is beyond question; and on the third day, after dinner, the depression nearly approached the neutral line. I do not lay any stress on the falling off in the *acidity per* 1000 *parts*, because the urine invariably became more aqueous after meals, and the falling off in the *degree* of acidity might seem attributable to this cause alone.

If we compare the hourly discharge of acid with the hourly discharge of solids, the depression of acidity after the meals, so faintly indicated in the above tables, comes out much more strongly, as will be shown hereafter.

To isolate more completely the operation of vegetable food, it was thought desirable to subsist for several days continuously on a purely vegetable diet, and to avoid especially taking supper on the previous nights. The articles of diet used were bread, rice, potatoes, carrots, lettuce and endive, with coffee and tea without cream. No alcoholic drinks were ever used during these experiments unless when specially mentioned.

TABLE XI. Vegetable food only. Mean of five days: three of these were consecutive, and the other two succeeded to days of mixed diet, on which dinner had been taken at Two p.m., after which no solid food was taken until next morning. Breakfast at Eight, dinner at Two.

Time of day	Hourly flow	Density	Solids per hour	Acidity		Alkalinity		Appearance	Remarks	Diet
				Per 1000	Per hour	Per 1000	Per hour			
7–8	274	1029·20	17·95	1·89	0·52	...	...	Clear, amber; nearly always depositing lithates on cooling.		Breakfast at 8. Coffee (with sugar and no cream) with bread.
8–9	480	1026·20	28·34	1·38	0·65	...	...	Clear, amber; often depositing lithates on cooling.		
9–10	920	1018·11	33·54	0·35	0·22	0·72	0·60	Three times clear, once turbid from phosphates.	Twice alkaline, once neutral, and once acid.	
10–11	820	1021·39	35·28	0·90	0·60	...	...	Clear, pale amber; sometimes depositing lithates.		
11–12	1670	1011·38	42·19	0·63	1·04	...	...	Clear, paler amber; not depositing at all.		
12–2	1780	1013·26	44·56	0·69	1·09	...	...	Clear, pale amber; not depositing at all.		
2–4	1630	1017·81	40·17	1·07	0·85	0·28	0·24	Clear, pale amber.	Four times acid, once alkaline.	Dinner at 2. Vegetable soup, salad, bread, potatoes, carrots, water.
4–6	1370	1016·20	39·31	0·72	0·37	0·50	1·14	Twice turbid from phosphates, three times clear.	Three times acid, twice alkaline.	
6–7	1250	1020·33	43·34	0·62	0·44	0·00	0·00	Clear, pale amber in each case.	Once neutral, twice acid.	
7–9	1340	1019·32	57·66	0·78	0·92	...	...	Clear, pale amber.		Two or three glasses of water between 7 and 11.
9–11	690	1022·93	36·57	1·30	0·92	...	...	Clear, amber.		
11–1	1020	1015·81	34·31	0·72	0·69	...	...	Clear, pale amber.		
1–7	360	1020·09	14·04	1·38	0·43	...	...	Clear, reddish amber.	(Asleep).	
7–8	280	?	?	1·63	0·42	...	...	Clear, rich amber.		

TABLE XII. exhibits the Variations in the Acidity for the four days composing Table XI.
(The Plus and Minus signs used as before).

Time of day	Feb. 4, First day		Feb. 5, Second day		Feb. 6, Third day		Feb. 8, Fourth day		March 10, Fifth day		Diet
	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	
7–8	+2·25	+0·70	+1·54	+0·53	+1·75	+0·42	...	...	+1·68	+0·44	Breakfast at 8. Coffee with sugar, but no cream, and bread.
8–9	+2·44	+1·00	+1·30	+0·79	+1·16	+0·53	...	...	+0·64	+0·28	
9–10	+0·70	+0·45	0·00	0·00	−0·36	−0·45	...	...	−1·08	0·75	
10–11	+1·54	+0·86	+0·80	+0·67	} +0·10	+0·17	} +0·80	?	+0·86	+0·26	
11–12	+0·76	+1·06	+0·73	+1·79					+0·42	+0·72	
12–2	+0·68	+1·33	+0·96	+1·29	+0·37	+1·08			+0·76	+0·68	
2–4	+1·60	+0·93	+1·20	+1·22	−0·28	−0·42	+0·54	+0·74	+0·94	+0·58	
4–6	+0·84	+0·37	−0·60	−0·81	−0·40	−1·47	+0·86	+0·58	*+0·46	+0·15	Dinner at 2. Vegetable soup, salad, bread, potatoes or carrots, water. Between 7 and 11 two or three cups of water, but no solids.
6–7	+0·96	+0·72	0·00	0·00	+0·50	+0·59	} +0·96	+1·09	+0·92	+0·60	
7–9	+0·80	+0·98	+0·44	+1·03	?	?			+0·90	+0·58	
9–11	+1·40	+1·12	+1·76	+1·19	+1·16	+1·07	+1·30	+0·64	+1·00	+0·50	
11–1	+0·76	+0·73	+0·88	+0·77	+0·90	+0·80	+0·34	+0·46	+1·42	+0·48	
1–7	+1·27	+0·50	?	?	+2·16	+0·40	+0·69	+0·40			
7–8	+1·54	+0·53	+1·75	+0·42	+2·53	+0·38	+1·25	+0·46	+1·00	+0·29	

* The urine from four to six was passed in two portions. That from four to five was alkaline to the degree of 0·30 per 1000, and 0·27 per hour; but between five and six it had become acid to the degree of 0·76 per 1000, and 0·42 per hour; so that the united products gave an acid reaction as represented in the table.

On the first day of this set the acidity fell in a very marked degree both after breakfast and dinner; on the second, the urine became neutral after breakfast for an hour, and alkaline and cloudy for two hours after dinner, and continued neutral yet another hour; on the third day it was alkaline after both meals; and on the fifth morning a breakfast of four pieces of dried toast and a cup of coffee without milk made the water passed two hours after turbid from phosphates, and highly alkaline; it lost its acid reaction after dinner also on this day. Here lies abundant evidence that vegetable food is able to depress the reaction of the urine equally with mixed diet.

3. *Effect of purely Animal Food.*

As in the case of vegetable food, two sets of experiments were made to ascertain the power of purely animal food to depress the acidity of the urine. The first set did not embrace the twenty-four hours, but began at seven in the morning and ceased about ten in the evening. Four days are included in this set; three of them alternated with days on which vegetable food only was taken at breakfast and dinner, but a supper of mixed food with ale or porter was taken at night.

TABLE XIII. Purely animal food. Mean of four days, not consecutive.
Breakfast at Eight, dinner at Four.

Time of day	Hourly flow	Density	Solids per hour	Acidity Per 1000	Acidity Per hour	Alkalinity Per 1000	Alkalinity Per hour	Appearance	Remarks	Diet
7–8	380	1029·80	21·69	2·52	0·88	...	...	Clear, amber; depositing lithates on cooling.		
8–9	720	1025·00	43·69	1·52	0·84	...	...	Clear, deep amber.		Breakfast at 8. Meat, eggs, milk and water.
9–10	1160	1023·75	61·06	0·64	1·03	0·38	0·26	Clear, amber.	Twice acid, twice alkaline.	
10–11	1110	1024·75	62·00	0·41	0·55	1·36	1·05	Three times clear, once turbid.	Three times acid, once alkaline.	
11–12	1030	1023·00	50·39	0·75	0·81	...	...	Clear, amber.		
12–2	940	1023·62	48·01	1·54	1·29	...	...	Clear, amber.		
2–4	860	1024·32	41·01	2·02	1·45	...	...	Clear, amber.		
4–5	740	1025·83	43·96	1·38	1·01	...	...	Clear, amber.		Dinner at 4. Meat, cheese, milk, water.
5–6	1770	1014·66	53·13	0·38	0·53	...	...	Clear, pale amber.	(Three days).	
6–7	1420	1022·00	70·44	...	...	0·59	0·90	Twice very turbid and yellowish, once clear.	(Three days).	
7–9	1640	1020·00	71·56	0·50	1·09	0·66	0·69	Twice turbid, once clear.	(Three days). Twice alkaline, once acid.	
9–10	1120	1026·50	69·30	0·67	0·74	...	...	Clear, amber.	(Two days).	

TABLE XIV. exhibits the Variations in the Acidity for the days composing Table XIII. (The Plus and Minus signs used as before).

Time of day	Jan. 24, First day		Jan. 26, Second day		Jan. 28, Third day		Jan. 19, Fourth day		Diet
	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	
7–8	+3·14	+0·77	+1·88	+1·15	+1·56	+0.78	+3·52	+0·84	Breakfast at 8.
8–9	+1·76	+0·68	+0·97	+1·07	+0·96	+0·91	+2·40	+0·66	Eggs and meat, with milk and water.
9–10	−0·32	−0·22	+0·68	+1·10	+0·60	+0·96	−0·44	−0·31	
10–11	+0·20	+0·14	+0·52	+0·68	+0·52	+0·84	−1·36	−1·05	
11–12	+0·88	+1·23	+0·70	+0·80	+0·96	+0·91	+0·48	+0·29	
12–2	+0·86	+1·19	+1·28	+1·11	+1·20	+1·24	+2·84	+1·62	
2–4	+0·98	+1·47	+2·00	+1·36	+1·74	+1·44	+3·30	+1·55	
4–5	+1·23	+0·99	+1·24	+0·94	+1·68	+1·10	+0·38	+0·68	Dinner at 4.
5–6	+0·90	+1·03	+0·10	+0·16	+0·16	+0·41	−0·62	−0·62	Meat, cheese, milk and water.
6–7	−0·15	−0·17	−0·88	−1·28	−0·73	−1·25	+1·58	+1·39	
7–8	} +05·1	+1·09	−0·94	−1·32	−0·95	−1·54	} +1·88	+1·93	
8–9			−0·14	−0·16	−0·62	−0·78			
9–10	...	...	+0·94	+1·03	+0·40	+0·46	...	...	
10–10½	...	...	...	...	+0·94	+ ·94	...	...	

The urine became alkaline each day after dinner, and twice after breakfast. On the third day, after breakfast, the urine sustained its acidity with a scarcely perceptible diminution; though, as compared with the height to which it rose after the supposed alkaline tide had passed off, the depression was sufficiently marked, and still more marked if the hourly excretion of solids be taken into account, at the contrasted periods.

In order to observe the effect of a longer continuance of an animal diet, the supper of mixed food on the previous night was discontinued, and for three successive days animal food alone was taken. The following tables exhibit the results obtained.

TABLE XV. Animal food only. Mean of four days, three of which were consecutive. Breakfast at Eight, dinner at Two.

Time of day	Hourly flow	Density	Solids per hour	Acidity Per 1000	Acidity Per hour	Alkalinity Per 1000	Alkalinity Per hour	Appearance	Remarks	Diet
7–8	270	?	?	2·40	0·62	...	...	Clear, amber.		
8–9	510	1027·00	33·60	1·66	0·82	...	...	Clear, amber.		Breakfast at 8. Meat and eggs, milk, water.
9–10	870	1020·50	47·52	0·23	0·19	0·20	0·39	Clear, amber.	Twice neutral, once acid and once alkaline.	
10–11	1570	1019·75	49·75	0·49	0·41	0·44	0·35	Three times clear, once cloudy.	Twice acid, once neutral and once alkaline.	
11–12	1470	1021·81	54·07	0·91	0·92	...	...	Clear, pale amber.		
12–2	950	1025·31	53·32	1·41	1·26	...	...	Clear, amber.		
2–4	940	1024·50	53·00	1·39	1·25	...	...	Clear, amber.		Dinner at 2. Meat, cheese, milk, water.
4–6	1300	1021·20	58·79	...	...	0·47	0·58	Muddy, yellowish.		
6–7	1110	1026·10	64·49	0·56	0·47	0·46	0·67	Twice muddy and yellowish, twice clear.	Twice acid, twice alkaline.	
7–9	1210	1021·00	61·01	0·81	0·96	...	...	Clear, amber.		
9–11	1080	1024·90	62·42	1·05	1·11	...	...	Clear, amber.		A glass of water between 9 and 11.
11–1	2150	1013·25	53·87	0·51	0·91	...	...	Clear, straw.		
1–7	480	1021·12	22·24	1·43	0·64	...	...	Clear, reddish amber.		
7–8	310	?	?	2·59	0·77	...	...	Clear, rich amber.		

TABLE XVI. exhibits the Variations in the Acidity for the several days composing Table XV.

Time of day	Feb. 6, First day		Feb. 7, Second day		Feb. 8, Third day		Feb. 9, Fourth day		Diet
	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	Per 1000	Per hour	
7–8	+1·25	+0·46	+2·53	+0·38	+3·68	+0·86	+2·13	+0·79	
8–9	+0·78	+0·48	+1·45	+0·58	+2·36	+1·10	+2·04	+1·12	Breakfast at 8.
9–10	−0·20	−0·39	0·00	+0·00	+0·46	+0·37	0·00	0·00	Meat or eggs,
10–11	0·00	0·00	−0·44	−0·35	+1·56	+1·22	+0·40	+0·43	or both, milk,
11–12	+0·22	+0·73	+0·80	+0·60	+1·28	+1·18	+1·37	+1·17	water.
12–2	+1·04	+1·33	+1·44	+1·03	+1·20	+1·34	+2·00	+1·33	
2–4	+0·46	+0·51	+1·80	+1·37	+1·50	+1·61	+1·80	+1·51	Dinner at 2.
4–6	−0·60	−0·83	−0·76	−0·81	−0·45	−0·42	−0·07	−0·27	Meat and milk,
6–7	−0·57	−0·97	−0·34	−0·36	+0·52	+0·52	+0·60	+0·42	cheese, water.
7–9	+0·14	+0·17	+1·14	+1·21	+0·96	+1·39	+1·00	+1·07	
9–11	+0·62	+0·76	+1·24	+1·24	+1·32	+1·27	+1·00	+1·15	A cup of water
11–1	+0·21	+0·49	+1·06	+1·18	+0·40	+1·14	+0·37	+0·85	between 9 and
1–7	+1·09	+0·58	+1·52	+0·75	+1·04	+0·61	+2·06	+0·64	11.
7–8	+2·25	+0·72	+3·68	+0·86	+2·13	+0·79	+2·30	+0·72	

The urine lost its acid reaction each day, both after breakfast and dinner, except on the third day, and even then it was reduced nearly to the neutral line.

The diet used in these two sets of observations was variously composed. For breakfast, a mutton or pork chop, or beef steak, and water; sometimes eggs and boiled milk, and once fried sole and boiled milk. For dinner, roast fowl, partridge or hare, broiled salmon, oysters, beef steak, mutton chop, cheese and milk.

On the Effect of Food in general.

Inasmuch as all our ordinary articles of diet, whether they be drawn from the vegetable or the animal kingdom, present the same elements of composition beneath a great diversity of outward condition, it might naturally be anticipated that their effects on the system would not be greatly dissimilar.

In every article of diet, as offered by the hand of nature — in the flesh of beasts, birds, fish, and all other forms of animal life; in the seed of the various orders of cereals; in the succulent stems, roots and tubers of fresh

vegetables; in the sweet and subacid fruits of our own and tropical climates — in all these, may be found representatives of the albuminous, oleaginous and saccharine groups of alimentary substances, together with certain saline ingredients — phosphates, sulphates, chlorides and carbonates, having for bases soda, potash, lime and magnesia — whose universal presence sufficiently attests the essential importance of their functions.

Nevertheless the wide differences of proportion which are known to exist in the admixture of the organic and inorganic substances in various articles of diet, and especially in the contrasted classes of animal and vegetable foods, prepare us to expect that in the final products of the vital operations there will be found certain peculiarities attributable to the nature of the aliment. One of these is the reaction of the urine, which is notoriously dissimilar in carnivorous and herbivorous creatures, being acid in the former and alkaline in the latter. And this difference has been universally laid to the account of the food of the two classes.

The urine of the herbivora is alkaline, it is asserted, because they feed upon matter rich in alkaline carbonates, citrates and tartrates, all of which appear in the urine as carbonates. And it has been shown that when these creatures are made to fast, their urine becomes acid.

Dr. Cl. Bernard* was able to trace still more decisively the connection between the reaction of the urine and the nature of the food. He found that when rabbits (whose urine is normally alkaline) were fed for some time on an exclusively animal diet, they passed an acid urine; and that its alkalinity was not restored until a vegetable diet was substituted. Dogs also, when restricted to a vegetable fare, secreted an alkaline urine, turbid from deposition of phosphates; but when restored to animal flesh

* *Comptes Rendus*, 1846, tom. 22, p. 534.

their urine resumed its natural clearness and acid reaction.

It is more easy to reconcile the experiments detailed in the preceding pages with the first of these considerations than with the second. All food was found essentially to affect the reaction of the urine alike, but, contrary to what one would expect, animal food produced usually a stronger and more enduring impression than vegetable food. Yet it may be pointed out as worth notice, that of the three *consecutive days* of exclusively animal and exclusively vegetable diet, the greatest effect in the former was on the first day, and it fell progressively on the second and third days; whereas the reverse took place on the days of vegetable food. On the first day the urine did not become alkaline at all; on the second it was neutral after breakfast and alkaline for two hours after dinner; on the third day it was strongly alkaline for an hour after breakfast and for three hours after dinner.

Ordinary food, whether it was purely animal, purely vegetable, or, as was more usual, an admixture of the two, was invariably found to cause a diminution in the amount of acid separated by the kidneys. In the tables there is record of thirty-two days on which the urine was examined both after breakfast and after dinner (with the exception of one day on which the observations did not commence until after dinner); and I have notes, in addition, of the state of the urine on five other days after breakfast and on four days after dinner. The following table shows how often the urine became neutral or alkaline, or sustained its acidity, after these two meals, with various kinds of food.

TABLE XVII. shows the reaction of the urine at the time of greatest depression after breakfast and dinner on mixed food (twenty days breakfast and nineteen days dinner); vegetable food (eight days breakfast and nine days dinner); and animal food (eight days breakfast and dinner).

	Mixed diet			Vegetable diet			Animal diet		
	Alk.	Neut.	Acid	Alk.	Neut.	Acid	Alk.	Neut.	Acid
Breakfast	12	2	6	2	1	5	3	2	3
Dinner...	17	2	0	3	0	6	8	0	0

Out of seventy-two meals the urine became alkaline after forty-five, neutral after seven, and sustained its acidity after twenty. After thirty-six breakfasts the urine became alkaline seventeen times, neutral five times, and remained acid fourteen times. After thirty-six dinners the urine became alkaline twenty-eight times, neutral twice, and it continued acid six times.

The effect of dinner is thus seen to be very considerably greater than that of breakfast. Indeed after a dinner of mixed or animal food the urine never failed to sink to the neutral line, its acidity being preserved only with vegetable food. The cause of the distinction lay, probably, simply in the fact, that breakfast was a much lighter meal than dinner, and its impression on the system consequently smaller.

But although the urine preserved its acidity frequently after breakfast, and sometimes even after dinner, there was a notable falling off in the intensity of its reaction, whether regard be had to the degree of acidity per 1000 parts, or the quantity discharged per hour. In one set of experiments only, namely the first on vegetable food, does this appear at first sight somewhat doubtful, and seem to require some additional explanation. The decline in the hourly discharge of acid after breakfast, as seen in Table IX., seems so small that a doubt might be cast on its reality; but if we compute the hourly separation of acid as it stands related to the hourly discharge of solids, we

shall find that the fall after breakfast is brought out in its true prominence, as the following table shows.

TABLE XVIII. The first column is a transcription of the hourly discharge of acid from Table IX. The second column shows the per-centage of acid on the solids at successive hours from seven a.m. to four p.m. (Breakfast at Eight).

Time of day	Acid discharged per hour	Acid corresponding to 100 grains of solids
7–8	1·20	3·87
8–9	1·35	2·87
9–10	1·14	2·11
10–11	1·00	1·72
11–12	1·19	2·12
12–2	1·32	2·32
2–4	1·30	3·82

By this table it is made evident that the solid constituents of the urine, or what, for brevity, may be called the *solid urine*, became steadily less and less acid after breakfast until eleven o'clock; from that time its acidity rose as steadily until dinner. At eleven o'clock the solid urine had less than one-half the acidity it possessed before breakfast, or just before dinner.

The same result is brought out after dinner, and in about the same degree. Taking the two hours before dinner, and the sixth and seventh hours after dinner, every 100 grains of solid residue had an acidity of 3·65; whereas during the third and fourth hours after dinner (the period of the supposed *alkaline tide*) the solid residue of the urine had but 1·82 per cent. of acid.

The apparently exceptional cases, where meals do not appear to lower the acidity at all, or where the hourly discharge rises even for a while after a meal, are thus made conformable to the general result. The taking of a meal greatly increases the excretion of solids by the kidneys; and even if these be of diminished acidity, the quantity passed per hour may overbalance this diminution, and, for a while, actually cause an increased hourly discharge

of acid. For example, on the 25th of January (Table X.), the hourly separation of acid before breakfast was 1·08. During the first hour after breakfast it rose to 1·25; but if the solid matters be brought into consideration, it is found that there is here in reality a depression, instead of an elevation, of the acidity. At the former period the solids had a per-centage of acid of 3·81; whereas at the latter, the per-centage had declined to 3·17.

I have been thus at pains to prove that the evidence of the first set of experiments on vegetable food is unequivocal as to the depressing effect of vegetable food on the reaction of the urine, not because other proof was wanting — the second set abundantly supplies that — but in order to uphold the conclusion contended for, that all the observations on the individual under experiment were perfectly concurrent; and that the law, so far as he was concerned, came out absolute and without exception, *that food lowered the acid reaction of the urine.*

There is considerable advantage in comparing the oscillations of the urinary free acid with the oscillations in the hourly discharge of solid urine. It is from this point of view that we can best see the relations of the former to the state of the blood. It is not necessary here to go into the proof that the degree of alkalinity of the blood regulates strictly the rising and falling acidity of the urine. By adding to the alkalescence of the blood through artificial means, as by exhibiting caustic or carbonated alkalies internally, we are able to depress in corresponding proportion the acidity of the urine. On the other hand, also, by exhibiting acid (although this seems less readily accomplished) we can similarly heighten the reaction.

By taking the solid urine as a standard of comparison, we avoid two fallacies which respectively affect the determinations per 1000 parts and the determinations per hour. We escape, in the first place, oscillations arising from *mere*

dilution, which sometimes sink the determinations per 1000 grains almost to zero, although the separation of acid at the time is high. In the second place, we avoid oscillations arising from the *varying activity of the kidneys*.

I shall presently have to call attention to the columns showing the hourly discharge of solid urine. It will be sufficient here to state, that, after meals, the hourly discharge of solids rises very considerably, that at periods remote from meals it sinks again, and that during sleep it falls to a minimum. So great is the oscillation from this sole cause that it gives an entirely false complexion to the columns indicating the acid separated per hour. It makes it appear as if the acidity, after having recovered from the first depression consequent on a meal, rose for a few hours to an unusual height, and then fell away again a second time — as if, in fact, an *acid tide* succeeded to the *alkaline tide* previously to the subsidence of the reaction to its normal level. If the eye be cast along the columns of hourly discharge of acid in the various tables, this will be seen at a glance. During the hours of continued abstinence after dinner it comes into especial prominence. It is seen that when the alkaline tide has subsided, and the acid reaction become re-established, there is exhibited for about two hours an unusually high rate of discharge; in fact higher than at any period of the twenty-four hours; and that after this again there appears a constant fall which is maintained and increased during the subsequent hours of abstinence and sleep. All the tables concurrently demonstrate this (see Tables I., III., XI. and XV.), and without the correction here indicated it would lead to an erroneous interpretation of facts.

Now if we take the *solid urine*, and calculate its acidity per 100 parts, we find that after recovering from the depression of the alkaline tide the acidity shows no sign of

falling off again until after the next meal. The determinations in Table III. are especially worthy of attention in reference to this point. This table gives the mean numbers for seven successive days, on which the utmost endeavour was made to avoid irregularity or inaccuracy; and the densities were all taken by the specific gravity bottle. If we now place, side by side, the numbers indicating the hourly discharge of acid and the numbers indicating the per-centage of acid in the solid urine, from the hour of seven p.m., when the alkaline tide subsided, to eight o'clock next morning, just before breakfast, the two series of numbers will be seen to be entirely different. In the former there is a rise and then a fall; in the latter there is a continued rise. For comparison, the acidity per 1000 grains of the liquid urine is also added.

TABLE XIX. shows the varying results obtained by computing the acidity of the urine in three different ways, namely, per 1000 *grains of the liquid urine; per hour; and per* 100 *grains of the solid urine. (The two first columns copied from Table III.) Dinner at Two p.m.*

Time of day	Per 1000 grains of liquid urine	Per hour	Per 100 grains of solid urine
7–9 p.m.	0·55	0·48	1·02
9–11 „	0·93	0·77	2·02
11–1 „	1·07	0·62	2·13
1–7 a.m.	1·30	0·38	2·37
7–8 „	1·55	0·45	2·50

From the third column, with which the first also is in agreement, we may conclude that when the alkalescence of the blood regains its level, after the influence of a meal has passed off, it continues nearly at that level until the next meal; or, rather, there appears a tendency to a gradual diminution in the alkalescence of the blood as fasting is prolonged; but I scarcely dare to rely with certainty on the calculation for the determination of so nice a point. The other tables of means, Nos. I., XI. and

XV., support the main conclusion as uniformly as could be expected, if it be remembered that the densities were taken by a hydrometer, without correction for temperature, and not with that care and precision which would have been used had it been foreseen that they would have been employed for calculating the solids.

The rise in the hourly discharge of acid from nine to eleven and eleven to twelve (eighth, ninth and tenth hours after dinner) is, therefore, entirely due to the fact that the increased activity of the kidneys, called forth by the meal, persists for two or three hours after the blood, and by consequence the urine, has recovered its normal reaction.

So that by taking the solid urine as a basis for calculation, two distinct corrections have been shown to be necessary in reading the determinations of acidity per hour. By the first correction, the apparently doubtful or contradictory cases, where there was but a slight or no fall in the numbers after breakfast, are made to agree satisfactorily with the general law; and by the second, the apparent existence of an acid tide following in a few hours on the ebb of the alkaline tide is shown to depend on a fallacy.

The duration and time-of-setting-in of the alkaline tide were both subject to considerable variations from day to day, and they differed too for breakfast and dinner. Under the term alkaline tide is embraced the whole period of depressed acidity, whether the urine at the time was alkaline, neutral or only of diminished acidity.*

The effect of breakfast appeared earlier than that of dinner, and was always distinctly perceptible at nine

* It might be objected to this term, *alkaline tide*, that it is applied to urines which do not become alkaline. But, although in such instances the urine does not altogether lose its acidity, there is only a difference of degree between such and those in which the urine becomes actually alkaline. There is the same movement in both cases; but in the former it does not crop out from beneath the surface of the neutral line, whereas in the latter it makes itself sensible by a change of reaction.

o'clock, that is, within forty minutes after the conclusion of the meal. The urine, however, never became alkaline, nor even neutral, so soon. During the succeeding hour, from nine to ten, the alkaline tide usually culminated; but in about a third of the cases the point of least acidity was not reached until eleven o'clock. Then the tide turned; and from eleven to twelve the urine was found fast recovering its reaction, and about one the normal level was generally attained.

But although the acidity after breakfast was *depressed* for a period of from four to five hours, it was not *absolutely alkaline* usually for more than one hour; generally from nine to ten. Not unfrequently, however, it continued alkaline, and even increased in alkalescence, until eleven; and on one occasion it continued alkaline for three hours, that is, until noon.

After dinner, which usually occupied half an hour, the acidity maintained itself through the first hour (that is, the hour at the beginning of which dinner was taken), but declined during the second in varying degrees on different days; and on five occasions the urine became alkaline. During the third, fourth and fifth hours the alkaline tide ran in its greatest strength. On the third and fourth hours the urine was always (with two exceptions) found alkaline when the meal had been of mixed food or animal diet. On the fifth hour it was also nearly always alkaline, but not so invariably so as on the two preceding ones. At the end of the next (sixth) hour the tide had generally turned and the acid reaction been restored. This change appeared often to take place with considerable suddenness, and the rise of the acidity went on with such celerity that in two hours (that is, at the end of the seventh hour) it had reached the ordinary standard. Three hours was the usual duration of the alkalescent state of the urine after dinner; sometimes two hours, more rarely four hours, and

one occasion five hours. The amount of free alkali hourly discharged after dinner was, generally, not far from double the quantity observed after breakfast; so that in duration and intensity the effect of dinner proved about twice as great as that of breakfast.

The Alkaline Urine of Food.

The alkaline urine which followed the taking of food deserves special description. It is important, in the first place, to state that its alkalescence did not depend upon ammonia, but upon a fixed alkali. It had no ammoniacal smell; caustic potash failed to evolve any; and the phosphates which were thrown down in it did not, when freshly passed, contain any crystals of the ammoniaco-magnesian phosphate. Under the microscope the fresh deposit was always found to be amorphous. No effervescence could ever be observed in the urine on the addition of hydrochloric acid. As might have been anticipated, the occurrence of an alkaline reaction determined the precipitation of the earthy phosphates, and the urine, when passed, was frequently turbid. But this was not always so. Not unfrequently, especially after breakfast, the urine, although alkaline, retained its transparency. Generally such a urine was of feeble alkalinity and dilute; but now and then it was observed to be tolerably concentrated, highly alkaline, and still clear. All transparent alkaline urines were rendered immediately turbid by caustic ammonia and by heating, so that the transparency did not depend on the absence of earthy phosphates. It was also found that, in the turbid urines after subsidence of their deposits, caustic ammonia caused an additional precipitation.

Out of twenty-eight specimens of alkaline urine after breakfast, nineteen were more or less turbid and nine clear. Of sixty-four specimens alkaline after dinner, fifty-one were turbid and thirteen clear. The proportion of

clear alkaline urines after breakfast was, therefore, considerably greater than after dinner.

The degree of turbidity varied from a barely perceptible cloudiness to a thick muddy opacity. The deposit subsided quickly and left a clear, yellowish-amber supernatant liquor, often with a greenish tinge.

The *odour* of this urine was peculiar, and so distinctive that its alkaline reaction could with certainty be predicated, without the aid of test paper, simply by the sense of smell. It was altogether devoid of the characteristic urinous odour, and exhaled a strong sweetish aroma resembling that of the fresh urine of the horse. The more strongly alkaline it was, the more powerful was this odour, and *vice versâ*.

The proportion of *alkaline and earthy phosphates* in the urine of the alkaline tide was found to be considerably increased. The quantity separated during the two hours preceding dinner and that separated during the third and fourth hours after dinner was subjected to comparison. The hourly separation of the *earthy* phosphates was found, on an average of six days, nearly doubled after dinner; and the *alkaline* phosphates rose from 3·47 grains to 4·90 grains per hour. This increase was not owing to the quickened activity of the secretory organs, for the proportion per 1000 of the liquid urine and the proportion per 100 grains of the solid urine exhibited an equally marked elevation.

The quantity of *uric acid* was ascertained for three periods on each of the seven days composing Table III., namely, in the urine of the alkaline tide after dinner (from four to seven); in the acid urine passed between nine and eleven; and lastly, in the urine of sleep. The average for the seven days, at each of these periods, may be seen from the following table.

TABLE XX. shows the mean amount of Uric Acid separated on the seven days composing Table III.—during the alkaline tide after dinner; from nine to eleven p.m.; and during the night.

Time	Uric acid per 1000 grains of liquid urine	Uric acid per hour	Uric acid per 100 grains of solid urine
4–7 p.m., alkaline tide	0·40	0·36	0·83
9–11 p.m., acidity restored	0·18	0·13	0·34
1–7 a.m., urine of sleep	0·39	0·10	0·60

The urine of the alkaline tide was, therefore, rich in uric acid (as calculated per 1000), rather more so even than the night urine, which in nearly every instance deposited urates copiously on cooling. The hourly quantity was almost three times greater than during the succeeding period when the acidity was restored (from nine to eleven), and more than three times greater than during sleep. The differences are not so great when the per-centage of uric acid in the solid residue is calculated; although here, also, the alkaline urine gives notably the highest figure.

The urine of the alkaline tide may, therefore, in every sense be regarded as pre-eminently charged with uric acid.

The remarkable poverty of the urine in uric acid, from nine to eleven, not only as contrasted with the urine of the alkaline tide but also with that of sleep, is probably closely connected with the increased excretion of uric acid during the preceding hours of the alkaline tide. If it be true, as these results lead us to believe, that a diminished acidity or, *a fortiori*, alkalescence of the urine is very favourable to the separation of uric acid from the blood, and causes an increased quantity of it to pass through the kidneys, it follows that on the cessation of the alkaline tide the blood must be unusually poor in uric acid; and the separation of it by the kidneys therefore, on the

re-establishment of the acid reaction, must be proportionally diminished. It is also extremely probable that there is, to some extent, an increased *production* of uric acid in the blood after food, and that a portion of the increased elimination is due to this.

Remote Effects of a Meal.

Although, as we have seen, the immediate effect of a meal was to depress the acidity of the urine, the more remote consequence was to uphold and even to increase the acidity. It has already been pointed out, that if we take the amount of acid separated per hour as our standard of comparison, the quantity discharged was greater during the period immediately following the re-establishment of the normal reaction than at any other period either previous or subsequent. And this has been explained to have depended on the high activity of the kidneys at that time. But there is another and still more remote effect of a meal, which comes out under different relations, and which is seen most distinctly, when a comparison is made between the acidity of the urine on mornings succeeding supperless nights and that of the urine on mornings following a hearty supper. In the former case, the mean hourly rate of acid discharged between seven and eight a.m. was only 0·51; in the latter it was 0·88, or nearly double. And not only was the hourly discharge thus increased, but even the degree of acidity per 1000 showed a slight rise — the mean numbers being for the mornings after supperless nights 1·83, and for mornings after a supper 2·15. This latter calculation, however, did not always exhibit results in accordance with this general mean, and exceptions occurred; but for the *hourly* calculation, all the separate results were consistent, sometimes in a greater, sometimes in a less degree, with the general mean.

It is important to bear these particulars in mind, for

they enable us to explain certain irregularities and discrepancies which appear when the effect of a meal on one day is compared with the effect of a similar meal on another day. For it frequently comes to pass that the remote effect of a previous meal interferes with, or completely masks the immediate effect of a succeeding one. Here, no doubt, lies the cause of the slight effect produced by vegetable food in the first set (Table IX). The remote effect of the highly animalised diet of the previous days masked the immediate effect of the meals on the days following.

It is a marked feature in these experiments, that, although there was the greatest constancy and certainty in the successive changes of reaction, there were very considerable — indeed the widest — differences in the *absolute amounts* of free acid or free alkali separated by the kidneys at corresponding hours after a meal on different days. For example, in Table IV., the acid passed between eight and nine a.m. on the first day was more than four times greater than on the third day at the same hour; and this, although the breakfast on the two days was as nearly as possible the same both in quantity and quality. Two considerations may be offered in explanation of this discrepancy. First, there was the supper on the night preceding the first day; secondly, digestion and absorption of the meal was probably going on more rapidly on the third morning than on the first, from the comparatively empty state of the vessels, consequent on the long previous abstinence.

But these explanations by no means meet all the cases of irregularity, as is shown by a comparison of the state of the urine from ten to eleven a.m. on the fourth and fifth days of the same table. On the fourth the urine was neutral at this period, whereas on the fifth it was acid to the extent of 0·20 per hour; yet on both days the antecedents, so far as food was concerned, were identical. Doubt-

less there are secret causes of unequal action in the animal system of so subtle a nature as to be altogether beyond our present powers of appreciation. I would suggest, however, that the state of the body in respect to repose and exercise, and the external temperature, probably exert an important influence.

Muscular juice is highly acid, and its quantity is, in all likelihood, greatly increased by exercise; and it seems not improbable, as Vogel has suggested, that the degree of acidity of the urine has some connection with the quantity of this acid of muscle thrown into the blood, and so through the kidneys out of the body.

The external temperature, too, by quickening or retarding the cutaneous transpiration and the respiratory function, may affect the amount of acid circulating and generated in the blood.

Differences in the nature of the food, especially as regards the proportion of earthy and alkaline phosphates, may almost with certainty be named as operative in causing differences in the effects of different meals on the reaction of the urine.

This and other matters connected with this part of the subject will come under discussion again when the inquiry is entered on — *Why* should a meal depress the acidity of the urine?

The amount of free acid separated in the course of the twenty-four hours was found, on an average of nineteen days, to be sufficient to neutralise 14·10 grains of dried carbonate of soda, or an average of 0·58 grains per hour. The maximum quantity was 22·34, and occurred under a purely animal diet; the minimum was 5·90 under a mixed diet. Some days were found exhibiting throughout a feeble acidity; others a high acidity, quite independently of the nature of the diet. The average amounts of acid per

twenty-four hours for the different sets were: mixed diet (Tables I. and III.), 14·21 and 10·30; purely vegetable diet (Table XI.), 15·36; and purely animal food (Table XV.), 18·03. These numbers show that the daily amount of acid eliminated by the kidneys was not much or uniformly affected by the nature of the food. The smallest numbers occurred under mixed food; but I am not disposed to attribute this to the nature of the food so much as to other circumstances, inasmuch as on some of the days of mixed food the acidity ruled unusually high, but the averages (especially of those of Table III.) were greatly reduced by one or two days of very low acidity.

The degree of acidity per 1000 *grains* is of more importance, practically, than the amount per hour, inasmuch as the occurrence of urinary deposits depends on it. If no liquids were taken, the degree of acidity after the passing away of the alkaline tide, gradually increased until food was again taken. The highest acidity, therefore, was always found after the longest fasting, or just before breakfast and dinner. Between seven and eight in the morning the urine was uniformly found excessively acid, invariably depositing abundance of urates on cooling. The night urine likewise, except when liquids were taken on going to bed, was highly acid and sedimentary. It may be easily conceived, therefore, that this is the period most favourable to the formation of renal and vesical concretions. The urine flows slowly aud rests for a lengthened period in the bladder, while its excessive acidity and concentration diminish its solvent powers over oxalate of lime, uric acid and the urates — the three substances most liable to unnatural precipitation.

When no liquids were taken before going to bed the urine of sleep had an acidity varying from 1·50 to 2·16 per 1000. In the morning, before breakfast the numbers ran from 1·50 to 2·80 and 3·00, rising on one occasion to 3·68

per 1000. The mean acidity, taking all the hours during which the urine flowed acid, was 1·13 per 1000; if the hours of sleep, the two hours before dinner and the hour before breakfast be excluded, the mean acidity for the remaining hours of the acid flow falls to about 0·80 per 1000.

'*The amount of free alkali passed in the twenty-four hours* varied according to the duration and intensity of the alkaline tide. On the nineteen days represented in Tables I., III., XI. and XV. the urine was alkaline, on an average, for more than three and half hours each day, and contained a quantity of free alkali equal in saturating power to 3·32 grains of dried carbonate of soda. The mean quantity discharged per twenty-four hours, with mixed food, was 4·14 grains for the first set and 4·72 grains for the second (Tables I. and III.); for vegetable food, 1·71 grains (Table XI.); and for animal food, 1·68 grains (Table XV.)

The numbers for the separate days ranged from zero to 9·35 grains, and great irregularities prevailed throughout, even when the days were consecutive and the diet the same.

It is to be remarked that the days of mixed diet showed a greater discharge of free alkali than those of purely animal or purely vegetable diet. This corresponds with the lower figure representing the free acid on those days.

The hourly discharge of free alkali oscillated from 0·00 to 2·60; and it usually ranged between 0·50 and 1·00.

The degree of alkalinity per 1000 *grains* varied also in the same manner. The highest grade observed was 4·12; but the usual numbers were from 0·40 to 1·60.

In comparing the *effects of the three different kinds of diet* it must be owned that the results are very contradictory. In the first set of experiments on vegetable food the urine did not once become alkaline (Table IX.) nor

even neutral. This must be attributed to the remote effect of the flesh meat diet used on the alternate days. In the second set (Table XI.), however, vegetable food was found to possess a great and apparently increasing power of depressing the acidity of the urine when persevered in for successive days. So that the daily average discharge of alkali for the four successive days of vegetable food was raised even above that for successive days of animal food—the mean total amount of free alkali per day in the former case being 1·71 grains and in the latter 1·68 grains. In the first set of observations on animal food, however, the mean daily discharge of free alkali was 2·17 grains.

But the effect of mixed food was found on an average considerably greater than that of purely vegetable or purely animal food, both in duration and intensity—the mean daily separation of free alkali being 4·14 grains and 4·72 grains respectively for the first and second sets. This contradictory result appears at present quite inexplicable, for there did not seem to be any difference in the rate of absorption nor in the quantity of the meals.

The degree of acidity before meals (in other words, the remote effect) was found greatest after animal food, while the difference between purely vegetable food and mixed food was not very considerable; the numbers on the whole, however, being favourable to vegetable food. In the subjoined table may be seen the degree of the acid reaction during the hours of fasting, before breakfast and dinner, with the three kinds of food.

TABLE XXI. exhibits the Acidity of the Urine before meals, with the different kinds of food.

	From 1 to 8 a.m. (Before breakfast)		12 noon to 2 p.m. (Before dinner)	
Mixed food	1·47 per 1000	0·53 per hour	0·85 per 1000	0·74 per hour
Vegetable food	1·52 „	0·42 „	1·01 „	1·20 „
Animal food	2·01 „	0·70 „	1·46 „	1·27 „

Before proceeding to discuss the explanation to be offered of the power of a meal to depress the acidity of the urine, attention must be called to the variation in the amount of solid matter — in other words, solid urine — secreted at different hours of the day. The numbers were obtained, as already explained, by calculation from the density. With the exception of the seven days included in Table III., the specific gravity was taken by a hydrometer, and not with that scrupulous care, nor under that correction which is required for calculating the solids. While therefore able to state that all the tables show results remarkably concurrent, I am inclined to place most reliance on the determinations in Table III., which were made with the necessary precautions. On passing the eye down the columns of solids in the six tables of means, it is seen that the quantity per hour began to increase within an hour after breakfast, that is, between eight and nine, and went on increasing rapidly until eleven. It remained stationary until about noon, and then began to fall, and continued to diminish until dinner (see especially Tables I., IX. and XIII., when dinner was at four), and even for an hour after. At the end of the second hour after dinner there was usually, but not always, a slight rise. During the third hour there was a very decided rise, which went on increasing until the fifth or sixth hour, about which time it reached the culminating point. From that time the discharge of solids fell gradually until bedtime, and sank to the lowest point during the hours of sleep. All the tables of means exhibit the same processsion of numbers, whatever the nature of the food might be. Exceptional instances occurred, it is true, here and there; but with so great a multiplicity of details it was impossible altogether to avoid errors, from faulty observation, calculation or record. The contradictory facts, however, were quite insignificant beside the overwhelming majority of concurrent observations.

Now, what does this increase in the solids of the urine after a meal indicate? It evidently marks the passage of digested food into the blood. We see reflected in the renal secretion, as in a mirror, the flow of aliment into the blood. First the small beginning, then the increase, then the full tide, followed by a gradual ebb as the last portions of the food continue to be taken up from the lower divisions of the alimentary canal; until at length, after all has been absorbed, the blood returns to its condition as before a meal.

No doubt the passage of food into the blood is not the only circumstance that causes an increased separation of solid urine. Active functional exercise, whether of muscle or brain, probably quickens the activity of the kidneys by throwing into the blood an increased quantity of effete materials — the products of the destructive metamorphosis of the muscular and nervous elements. The invariably great and sudden rise in the solid urine from eight to nine in the morning is certainly not entirely the effect of breakfast, as it is incredible that digestion and absorption should have made such progress in forty minutes. The awakening of the whole system from the torpor of sleep — the aroused mental and muscular activity, and the consentaneous acceleration of the circulation and respiration — contributed, unquestionably, in producing the result. No such sudden rise is seen at any time in the subsequent hours, nor after dinner.*

The question now arises: How far is the depressing action of food on the acidity of the human urine a universal phenomenon? It is impossible to solve this question

* It may be noted that the increase in the hourly secretion of solid urine after dinner was not so prominently marked as after breakfast. I believe this arose from the almost complete inactivity usually observed by the subject of experiment for two or three hours after the former meal, whereas after breakfast his condition was one of activity.

except by greatly multiplied observations conducted on the principle of hourly determinations. But the facts at present known warrant the statement, that at least some healthy persons, whose urine presents no appreciable peculiarity, suffer constantly a diminished acidity of urine after meals, and that the secretion may flow alkaline and turbid for four and six hours daily without the least evidence of impaired function.

I am disposed to believe that the depression of the acidity of the urine after a meal is of universal occurrence; but, at the same time, observation leads me to conclude that there are very great differences in the degree or intensity of the phenomenon in different individuals — these differences arising, apparently, from fixed peculiarities of constitution. Without pretending to be able to point out all of these, or to estimate their influence, the following may be indicated as probably tending to produce that effect.

1. The cutaneous transpiration varies greatly in different persons. Those with an active skin eliminate a large amount of acid by the cutaneous surface, and consequently have less for removal by the kidneys. In such persons the alkaline tide might be expected to flow with greater intensity than where the skin is habitually dry and torpid. The sweat seems incapable of ever changing from acid to alkaline, for I have invariably found it sour, even when the urine has been rendered strongly and uninterruptedly alkaline by bicarbonate of potash for four and five weeks continuously.

2. The respiratory capacity of individuals varies. This bears on the present question in this way. Recent observations, especially those of Dr. E. Smith,* show that shortly after a meal the respiratory function is materially

* See his Papers read before the Royal and Medico-Chirurgical Societies during the present year, 1859.

quickened; more carbonic acid is exhaled, and consequently more oxygen absorbed into the blood. Those persons in whom this increase takes place promptly after a meal, and is unusually great, would probably show but a feeble alkaline tide, because the oxygen, thus thrown in excessive quantity into the blood, would cause increased formation of acid, and in this way mask the contrary effect of the food.

3. The quicker the digestion and the absorption of a meal, the greater *cæteris paribus* would be its depressing effect on the acidity of the urine.

It seems not unlikely that the two last mentioned causes determine by their mutual relation in an especial manner the degree of intensity assumed by the alkaline tide. A rapid absorption of a meal and a small amount of respiratory acceleration would present the combination most favourable to an intense depression of the urinary acid. On the contrary, a slow absorption of a meal and a prompt exaltation in the respiratory functions would so balance the opposing tendencies that the reaction of the urine would suffer only a minimum of depression.

These modifying circumstances apply not only to differences between individuals, but, in a minor degree also, to differences in the same individual at different times.

This seems the proper place to offer some considerations which may explain why alkaline urine after meals is not oftener met with in actual experience; and whence arise the grave discrepancies in the experience of different observers in this country and in Germany.

It is essential, in order to trace the effect of a meal, *to examine the urine at short intervals.* For if this be neglected the acid product secreted before and after the period of depression becomes mixed in the bladder with the urine of the alkaline tide, and when the whole is ejected by micturition it is found acid, even although a

portion of it was highly alkaline as it left the pelvis of the kidney. And micturition seems to occur, in the usual course, at such intervals as most effectually to prevent the alkaline urine of food from being observed unless by an unusual accident. The first micturition of the day usually takes place on leaving bed, and the urine is highly acid. The second does not occur, unless the bowels be emptied after breakfast, for some five or six hours; and it includes some very acid urine secreted before and immediately after breakfast, together with the urine of the morning alkaline tide, as well as the secretion with recovered acidity which is produced for two or three hours after the subsidence of the alkaline tide. Such a urine is sure to be acid, notwithstanding an alkaline flow through the kidneys for one or two hours. The same thing occurs after dinner. The bladder is usually emptied before the meal, and then for a couple of hours the urine flows acid. If there be excessive potation the urine may require discharging at this period, and it will always be found acid. If, on the contrary, potation has been restricted, or the system drained of water at the time of the meal, the next micturition may be delayed an hour or two, so as to cut the alkaline tide in half. Even then the urine will be acid, unless the depression was unusually great. The next micturition may occur after tea, and the recovered acidity of the urine would then conceal the alkalinity of the portion secreted at the beginning of the period.

From these remarks it may be gathered that unless the product of the alkaline tide be isolated, by emptying the bladder before and after its flow, no reliance can be placed on observations concerning it. This is, I apprehend, the reason why the observations of Beneke and Vogel, and probably also those of Dr. Sellers, have failed to support the conclusions of Dr. Bence Jones.

In Dr. Beneke's experiments the bladder was not emptied

oftener than about five times in the twenty-four hours; and by comparing the times of the meals with the times of micturition, it becomes evident that it was impossible for him to obtain results other than nugatory. The urines he examined were mixed urines, and he did not in any wise isolate the secretion at the critical periods.

The same objection applies to the observations conducted under the supervision of Vogel. The urines were collected during three periods—namely between breakfast and dinner (morning urine), between dinner and evening (afternoon urine), and during the hours of night. All such urines would be acid mixtures; but it by no means follows that they did not pass through an alkaline state, possibly even of some hours duration.

So true is it that the existence of the alkaline tide may be concealed for an indefinite time, even from those who are in the constant habit of observing the state of the urine, unless the urinary product be, as it were, analysed by frequent micturitions, that the urine of the subject of these experiments, though under close observation for some years, was not once known to have departed from its usual acid reaction; and it took me almost by surprise to find the phenomena of the alkaline tide so strongly and so remarkably pronounced after such long and effectual concealment.

A second circumstance, which must be borne in mind, is the *remote* influence of meals. As already fully explained, this remote effect of a previous meal frequently altogether masks the immediate effect of a recent one; and this statement seems especially to apply to the effect of supper on the succeeding breakfast. For this reason it is well, in order to obtain distinct results, to fast for eight or ten hours, or more if night intervene, before taking the meal whose effect it is wished to observe.

Finally, we come to the inquiry: *Why should a meal*

depress the acidity of the urine? Dr. Bence Jones imputes it to the diversion of the acid present in the blood to the stomach for the purposes of digestion. When the stomach is empty its lining membrane is neutral, or nearly so, and the acid generated in, or thrown into, the blood passes off by the skin and kidneys, rendering the urine acid; but when food is taken, acid gastric juice is poured from the blood vessels into the stomach, and the alkalinity of the blood is consequently raised. This causes the kidneys to separate a less acid or even an alkaline product; but when digestion is completed, the gastric acid returns to the blood with the chyle, and the urine regains its normal reaction. So that the stomach and kidneys are antagonistic in their reaction, the former being least acid when the latter are most so, and *vice versâ*. So plausible an explanation was at once adopted; and it is difficult to imagine that it may not have some operation in the way supposed, though, as I believe, it must be rejected as the main cause of the alkaline tide.

From Dr. Beaumont's twenty-fifth and twenty-sixth experiments, second series, we are led to conclude that sufficient gastric juice for the digestion of a meal is poured into the stomach within fifteen or twenty minutes after its ingestion; and that in half an hour, or an hour at most, the flow of acid into the stomach has ceased and absorption of the digested meal commenced. If this be so, the reaction of the urine ought to be most depressed within an hour or an hour and a half after a meal, instead of from two to five hours after. Moreover, how can it be explained, on this supposition, that the alkaline tide after dinner sets-in an hour or an hour and a half later than after breakfast?

But, whatever may be thought of the validity of these objections, is there not another solution more consonant with the facts observed? Is the alkaline tide not the

effect of the *absorption of a meal into the blood*, rather than of digestion? If it be true, as Liebig maintains, that the alkalescence of the blood — and in all animals that possess blood its reaction is alkaline — depends simply on the chemical composition of the alimentary substances, is there not here a solution for our question?

In his twenty-eighth Letter,* Liebig points out that phosphoric acid and the alkalies are present in such proportion in bread, meat, and our ordinary food, that if we suppose them dissolved the alkalies invariably preponderate. Hence arises, he says, the alkalinity of the blood. If this be so, every meal that is dissolved and absorbed into the blood must increase the alkaline reaction of that fluid and raise it for a time above the natural level.

But it is well known that when salts of the fixed alkalies which have an alkaline reaction — such as carbonates, basic phosphates and borates, or vegetable salts, which become carbonates in the system — are artificially exhibited, they change the reaction of the urine from acid to alkaline; evidently from inducing an excessive alkalescence of the blood, which it is the function of the kidneys to diminish by allowing the excess to escape in the urine. Conformably to this hypothesis, the earthy and alkaline phosphates were found greatly increased in the urine after meals.

A meal, therefore, viewed in this light, is a dose of alkali, which, when digested and absorbed, necessarily adds to the alkalinity of the blood; and, as a more remote but equally inevitable consequence, lowers the acidity of the urine or, if in sufficient quantity, renders it actually alkaline. It has been already pointed out that the setting-in of the alkaline tide coincides, in point of time, with the passage of the digested food into the blood, as indicated by the increased amount of solid urine secreted

* *Familiar Letters.* Letter 28.

by the kidneys; and as absorption goes forward and increases, the acidity of the urine diminishes more and more. It is nevertheless true that the subsidence of the alkaline tide is not synchronous with the cessation of absorption, for we have seen that the passage of food into the blood appears at its highest activity when the alkaline tide is beginning to ebb. This want of coincidence appears, *primâ facie*, to militate against the solution here offered; but it may be explained in two ways. Either it arises from the phosphatic salts being absorbed with more celerity than the rest of the food, and producing their effect before the other materials are all taken up; or, more probably, it depends on the increased absorption of oxygen by respiration, already noticed as occurring after a meal, which, after the lapse of five or six hours, by generating acid, counteracts the contrary effect of the food — in other words, from the remote effect of a meal overlapping the immediate effect.

If this solution be admitted, it brings all ordinary food into the same category with sub-acid fruits, which have long been acknowledged to possess the power, in virtue of their saline constituents, of rendering the urine alkaline; the only difference being that in the latter the effect is produced by salts, which become carbonates in the blood, and in the former by basic phosphates, which pass as such into the urine.*

* It must not for a moment be supposed that the urine is never alkaline (from fixed alkali) except after food. The urine not unfrequently loses its acid reaction in disease, independently of food, and presents to all appearance the characters of the urine of the alkaline tide after a meal. I have observed this character of the secretion repeatedly in the debilitated and anæmic condition which sometimes follows chronic subacute gout and obstinate subacute articular rheumatism; also in the course of some other protracted and exhaustive diseases that induce a chlorotic or spanæmic state; of which perhaps the most common is a certain form of atonic dyspepsia. In such cases the urine may be alkaline all the day through, though

The results of the foregoing observations may be summed up in the following propositions: —

1. The immediate and primary effect of a meal, whether of purely vegetable, purely animal, or mixed food, was, in from one to three hours to diminish the acidity of the urine; and very frequently to render it alkaline. The term "alkaline tide" is suggested to designate the period of depressed acidity.
2. The remote or secondary effect of a meal was to uphold and increase the acidity of the urine. This effect of a meal was especially observed over-night after supper.
3. The remote effect of animal diet appeared considerably greater than that of vegetable food. So that a highly animalised diet tends in the long run to heighten the acidity of the urine.
4. After breakfast, the greatest depression occurred at the second hour; and the period of depression continued from two to four hours.
5. After dinner, the greatest depression occurred at

this is very rare. In my experience I have not found such a condition continuing from day to day for any length of time, but rapidly passing away, not lasting more than some hours or a day or two; but perhaps returning again and again. Such urines must be carefully distinguished from ammoniacal urines, which invariably indicate some disorder in the urinary passages, generally in the bladder.

Another cause of alkaline urine is the immersion of the body in water. Homolle and Duriau found that after a bath the urine always lost its acid reaction, even when nitric acid had been added to the bath.—*Archiv. Génér.*, T. ii. 1856.

I am also convinced that an overworked and depressed state of the system promotes a diminished acidity of the urine; and that a high state of health and vigour tends to heightened acidity.

The explanation offered in the text, therefore, applies only to one particular case of alkaline urine, and must not by any means be taken as a general explanation.

the third, fourth and fifth hours, and lasted from four to six hours. The effect of dinner was greater, as well as more prolonged, than that of breakfast.

6. The effect of mixed and purely animal diet seemed almost identical. Vegetable diet, when used on alternate days with mixed or animal food, had a decidedly feebler effect; but when used continuously for several days successively its effect was equally powerful.
7. Alkaline urine after a meal owed its reaction to a fixed alkali. It was generally, but not always, turbid, when passed, from precipitated phosphates. Its odour resembled that of the fresh urine of the horse. It was richer in uric acid and in earthy and alkaline phosphates than the urine of fasting.
8. The depression of the acidity after a meal coincided in point of time with absorption rather than with digestion. The solids of the urine began to increase simultaneously with the declension of its acidity. So that the passage of food into the blood and the diminished acidity of the urine seemed to be connected together as cause and effect.

The following deductions appear also to be warranted: —

1. That the power of a meal to depress the acidity of the urine depends on its mineral constituents. These contain phosphoric acid and the alkalies in such proportion, that if we suppose them dissolved the alkalies invariably preponderate. Hence arises the alkalinity of the blood. If this be so, every meal that is dissolved and absorbed into the blood must for the time raise the alkalescence of that fluid above the natural level.

2. But the kidneys have the special function of regulating the degree of alkalescence of the blood. When it is too high they separate alkali, and the urine becomes alkaline; when it tends to become too low, on the other hand, they separate acid, and this gives to the urine its common acid reaction.
3. A meal then, in so far as its mineral ingredients are concerned, is but a dose of alkali, and its absorption causes, like any other dose of alkali, a depression of the acidity of the urine.
4. The emission of urine turbid with phosphates is, within certain limits, a natural phenomenon; and earthy phosphates constitute the only urinary deposit which can appear in the healthy urine *on passing*.
5. Urines may be divided into two chief classes: — First, *urines of fasting* (*urinæ sanguinis*); secondly, *urines of food* (*urinæ cibi*). Fasting urines are scanty and of high acidity; they present only one variety, namely, that of sleep, which differs from other fasting urines in possessing more colouring matter. The urines of food fall naturally into two divisions — those with a diminished, and those with a restored acidity; they are abundant in quantity. The urines of either class may be concentrated or dilute, according to the relation between potation and the requirements of the system in regard to water. So that *urinæ potûs* do not merit to be regarded as a distinct class. Most commonly the urine of micturition belongs exclusively to no division, but is a mixture of several kinds.

APPENDIX.

On two days the effects of cane sugar and honey were tried. Neither seemed to produce any depression of the hourly quantity of urinary acid discharged. On the first day a quarter of a pound of loaf sugar was dissolved in water and taken at eight a.m. At two p.m. half a pound was taken in the same way, instead of dinner. No solid of any sort, or other liquid than water, was taken during the experiment.

On the second day half a pound of honey was taken for breakfast at eight in the morning, with water; and no solid food was again taken until the conclusion of the experiment.

The annexed tables exhibit the results obtained.

TABLE I. Cane sugar. A slight supper taken the night before.

Time of day	Hourly flow	Density	Acidity Per 1000	Acidity Per hour	Appearance	Diet
7–8	285	1027·80	1·00	0·29	Clear, amber; depositing lithates on cooling.	Quarter of a pound of loaf sugar, and water at 8.
8–9	540	1025·16	0·92	0·46	Clear, amber.	
9–10	630	1022·00	0·68	0·43	Clear, amber.	
10–11	1730	1007·36	0·31	0·54	Clear, pale straw.	
11–12	2270	1006·24	0·18	0·41	Clear, pale straw,	
12–2	495	1021·24	1·12	0·55	Clear, rich amber.	
2–3	1530	1005·54	0·39	0·59	Clear, straw.	Half a pound of sugar, and water at 2.
3–4	*7780	1000·60	0·07	0·53	Colourless.	
4–6	1700	1006·84	0·32	0·54	Clear, pale straw.	
6–7	550	1026·32	1·00	0·55	Clear, deep reddish-amber; depositing lithates on standing.	
7–9	240	1029·36	2·36	0·56	Clear, deep reddish-amber; depositing lithates on standing.	
9–11	166	1030·32	2·16	0·36	Clear, deep reddish-amber; depositing lithates on standing.	

TABLE II. Honey. No supper the night before.

Time of day	Hourly flow	Density	Acidity Per 1000	Acidity Per hour	Appearance	Diet
7–8	388	1024·64	1·48	0·57	Clear, amber; depositing lithates on cooling.	Half a pound of honey with water at 8.
8–9	387	1027·84	1·72	0·67	Clear, amber; depositing lithates on cooling.	
9–10	455	1027·28	1·30	0·60	Clear, amber; depositing lithates on cooling.	
10–11	730	1015·52	0·68	0·50	Clear, amber; not depositing.	
11–2	660	1016·24	0·71	0·47	Clear, amber; not depositing.	
2–4	1760	1004·66	0·26	0·46	Clear, pale amber; not depositing.	
4–6	480	1016·44	0·70	0·34	Clear, pale amber; not depositing.	
6–7½	284	1025·76	1·22	0·35	Clear, rich amber; depositing lithates abundantly.	

* This was really the product of forty minutes.

Cane sugar and honey do not seem to possess any power of diminishing the excretion of acid by the kidneys. The evidence is rather the other way; for although the tendency to a rise after breakfast on the two days is too slight to insist much upon, yet it is very notable that the honey and sugar prevented the fall that would otherwise have taken place from mere prolongation of the fasting. The hourly flow kept up without appreciable change until about the seventh hour after the saccharine meal; then there occurred a marked declension.

The absorption of the sugar and honey was far from rapid; for sweet eructations took place for four and five hours after their ingestion, showing that the stomach was still hampered with their presence. Some considerable difficulty was experienced in swallowing and keeping down these large doses. It is worthy of note, that the second dose of cane sugar produced a most copious, though evanescent, diuresis. Not less than 7780 grain-measures of a watery urine were produced in forty minutes, or at the rate of 26½ fluid ounces per hour, presenting a remarkable contrast to the flow five hours later, when only 166 grain-measures were secreted per hour, or at a rate *seventy times* slower than during the first period.

Appendix.

PROCEEDINGS OF THE MICROSCOPICAL SECTION OF THE LITERARY AND PHILOSOPHICAL SOCIETY.

ESTABLISHED 1859.

INAUGURAL ADDRESS.

By Professor W. C. Williamson, F.R.S.,
President of the Section.

Printed from a Short-hand Report by John Slagg, Esq.,
the Secretary of the Section.

Gentlemen,—I will not waste your time in listening to apologies for the position I occupy to night; sufficient that the request of your Council to inaugurate your proceedings by an Introductory Address, was one that I did not feel justified in declining; the more so since your object is one in which I naturally feel a deep interest.

I have hailed with much pleasure the formation of a Microscopic Society; and I think we have wisely determined that it shall be a section of a still older institution, which I hope it will aid in antagonizing those materialistic tendencies that abound in this mercantile community. Hitherto the devotion of all our energies to the accumulation of wealth and material comforts has been too prominent a characteristic of Manchester society, to an undue exclusion of more elevating and soul-inspiring objects. Where such tendencies increase, and continue dominant, either in nations or individuals, we cannot be at a loss to foretel the issue. All former examples recorded in the world's history have shewn that as soon as a people make wealth and luxury their sole object of pursuit, the result is rottenness, and their ultimate fall merely a question of time. Any project, therefore, calculated to antagonize such tendencies cannot be received without extreme satis-

faction by thoughtful men. Let me not be misunderstood here: far be it from me to underrate the value of God's substantial blessings; they are amongst his choicest gifts to man, and properly used are designed to make him happy whilst they are employed for his Creator's glory. The pursuits in which you intend to engage have no such material purpose; on the contrary, by filling the mind with less selfish aims and loftier conceptions, they are calculated to bring man nearer to his Maker, and help to fit him for the position he is finally destined to occupy. I trust that the members of this Society, shunning high sounding titles and airy assumptions, will continually keep the highest objects in view; so that whilst they are lowly in profession they may be lofty in their aspirations.

There appear to be three definite purposes which may have influenced us in the establishment of this section: 1st, Amusement; 2nd, Self-instruction; 3rd, Instruction of others and the enlargement of the common stock of knowledge. The first of these objects is the lowest, but still a legitimate one. No thinking man can witness without pain the frivolity of the amusements popular even in fashionable society at the present day; many of them are unworthy of dignified manhood; consequently an important advantage is gained if we can substitute something which, whilst amusing, tends also to elevate and instruct.

That the second object is still a higher one needs no demonstration; but the third purpose to which I have referred merits a moment's attention. It behoves every thoughtful man to feel himself responsible for helping to leave the world wiser and better than when he entered it. On questions of religion and morality God has made to man a direct revelation of His will; but on questions of science He has made no such revelation: the solution of these He has intrusted wholly to human instrumentality.

But He has indicated His will, on this subject, by planting in man's bosom an intuitive craving for knowledge, which impels him to investigation and research, whilst it enables him to receive pleasure from those lofty generalizations that science finally aims to establish. If each man who, from the beginning, ever entered upon scientific inquiries, had but left behind him the trustworthy record of one single and previously unknown fact, how much larger a mass of information should we have possessed than at present?

I trust that our members may record many such facts; for the field of operation is so boundless, that steady perseverance is certain to be rewarded with success. In one respect we labour under a disadvantage compared with earlier observers. In the days of Leeuwenhoek and Hooke, when the microscope was a new instrument, and its revelations an unknown world, the observations made even by means of their imperfect instruments were wondrous, and their results great. They could not possibly go wrong in the search for subjects. They lay strewed around them on every hand. Consequently, we find the Memoirs of Leeuwenhoek abounding in notices of the structure of flies and spiders, sections of wood and such like matters, now familiar to every tyro. Not only was new material abundant, but his whole time could be given to its examination. Ploughing in a virgin soil, he was under no bondage to past and contemporaneous journalism — not troubled with the study of other men's researches, or in danger of being charged with appropriating other men's discoveries. With us the case is different. All common objects have been examined again and again. It is true a new field has been opened to us by the discovery of the achromatic lens; but the observations to which this improvement has led have assumed a more minute and abstruse character than was formerly

necessary, whilst our time for original inquiry is more limited by the necessity of reading and searching out what has already been done by other men.

Microscopic inquiry involves two processes which are essentially distinct, namely, the observation of facts, and their interpretation. I would earnestly recommend you to begin by disciplining yourselves in the observation of facts, ever honestly recording what you see. This is not by any means so easy a task as may at first sight appear; for we frequently find ourselves at a loss to distinguish facts from mere appearances. An illustration of my meaning presents itself in the case of the Naviculæ: How long were microscopists disagreed as to whether the dotted lines sculpturing their siliceous cases were due to elevations or depressions? Could the microscope but be made stereoscopic, much of this difficulty would be removed. In estimating the forms and relative positions of objects in a surrounding landscape, man derives his conclusions from two sources — from the indications afforded by linear and aerial perspective, and from the adjustment of the visual angle to objects at various distances. In working with the microscope, all the aids derivable from perspective are unavailing, from the nearness of the object and the almost uniform plane of all that can be brought within the focus of the microscope; and as the instrument is essentially monocular, we are equally deprived of the aid which the visual angle would afford us; hence, being deprived of both our ordinary means of measuring distances, we are compelled to fall back on more doubtful methods of estimating them. This circumstance demands that the microscopist should be strictly conscientious; that he should never assert anything to be a fact, whilst the slightest doubt remains in his mind respecting it. By so doing he would save much needless labour to others, who now have to go over the same ground, and furnish their confirmation, before the

supposed fact can take its place amongst the indisputable records of scientific progress.

But however difficult may be the observation of histological facts, their interpretation is still more so. As the former inquiry regards the *structure* of objects, the latter has mainly reference to their *functions*. Now observers have rarely, if ever, the means of watching the objects they are studying in all the phases of their vital operations; hence, to a logical mind, much collateral information becomes essential to the comprehension of what is seen. It was here that Ehrenberg, the most justly distinguished of microscopic investigators, so signally failed. He observed facts of structure accurately and well, but his mind was not adapted to their interpretation; hence the conclusions he arrived at, respecting the functions of organs, were rarely such as can now be accepted.

The essential characteristic of a truly philosophical observer lies in his power of inducing from an aggregation of facts the relations which subsist between them. No fact either stands alone, or is the result of accident; and when these relationships are ascertained and expressed, in some brief but comprehensive formula, we obtain what is called a law. To discover laws is the legitimate ambition of the highest intellects, as the undying distinction which such discoveries afford is one of its loftiest rewards; but so great is the labour and difficulty involved in the discovery of one such law, that it is only given to the loftiest intellects to wear the laurel-wreath. But let it be the ambition of each one of you to aim at the achievement of some such result.

I am anxious that this address should not deal exclusively with general topics, but that it should assume a practical character, calculated to aid you in your future proceedings. The course that I should most strongly recommend to any one just entering upon these pursuits, is

to select some standard work, such as those of Schwann or Schleiden, on cell-growths, or Balfour's *Manual of Botany;* in which the illustrations are mainly taken from common objects, easily obtained. I would recommend you to study these works with the microscope by your side, and, as you read, examine the objects which the authors describe. By thus repeating the experiments of others, on any subject, you gradually learn *what* to see and *how* to see, and are thus made familiar with the modes of conducting original researches. Whilst carrying on this process, you either confirm your author's observations, or note the points in which they do not harmonise with your own. When the latter is the case, the discrepancy suggests new observations, correcting or confirming your independent opinion; and you thus take your first steps as original observers.

Above all, I would guard you against desultory observations. Let whatever you do with the microscope be part of a fixed plan. The reward which, sooner or later, such a system brings will amply repay you for your labour.

Another course which I would strongly recommend to our younger members is, that each should undertake to lay before the Society a report on the existing state of knowledge on some fixed subject, accompanied by illustrative specimens. The twofold result would be the production of good and interesting Papers, and the acquirement by the reporter of that practical power of observation that would best fit him for original investigations. But, whilst adopting this plan, I cannot too strongly warn you against an uninquiring acceptance of all published results, and undue submission to the authority of great names. Aware beforehand of what we may expect to see, there is danger of our imagining that we do see all that has been described. I have myself been led astray in this manner, so that I speak feelingly. When I began my researches on the

Foraminifera, I was strongly influenced by Ehrenberg's opinion that they were Bryozoa, and gave countenance to that opinion in a Paper on the "Levant Mud," published in the *Transactions of the Manchester Literary and Philosophical Society*. Thus I was, for some time at least, the means of diffusing an erroneous idea. On the other hand, a youthful, overweening confidence, and a deficiency in modesty and merited respect for truly great men, cannot be too strongly reprehended. Try to hit the golden mean.

One of the most important studies that can claim your attention is that of the mysterious principle of life. Far be it from me unduly to check the aspirations of any philosophic student; but it may save us from uselessly expending our strength, to remember that there are some subjects which must be studied under limitations; subjects which, whilst the phenomena they present demand the devotion of our best energies, we may never hope, at least in this world, fully to comprehend their cause. Life is one of these subjects. We cannot apprehend it; we can only study its functions. Its source lies beyond the field of our mortal experience, and, probably, we shall never know more of its primary nature than we do at present. But the microscopist has already done much to remove misapprehension respecting it, and will do more. When Leeuwenhoek commenced his researches on this subject, he found it loaded with absurdities, most of those who preceded him, from Aristotle to Philippo Bonani, believed in spontaneous generation; even in his day this idea was not only applied to the obscure lower animals, but to those of comparatively higher organization, such as shell-fish and corn-weevils; and Leeuwenhoek found himself under the necessity of refuting such absurdities, by demonstrating that these objects were produced, like other animals, by the ordinary modes of generation. Driven from these positions, men were still unwilling to abandon the idea,

and took refuge amongst the less perfectly understood creatures, such as the Entozoa and infusorial Monads; but even here we have seen the error dispelled by the increasing light of science. Siebold has exploded it in the case of the Tape-worms, and other writers have done the same for the Infusoria; so that the fallacy is now left without a refuge. Greatly is the world indebted to science for this work, remembering what is the question involved. Let it be admitted that one of these lowest living creatures comes into existence independently of a Creator, and wherefore should a Creator be needful to man himself!

Though scarcely credible, there still exists a class of men who doubt the interpretations of the microscrope and distrust its revelations. In his Socratic dialogues, Plato makes Meno compare Socrates, who was constantly infusing doubts into others, to a torpedo, which benumbs whoever touches it. We occasionally meet with anti-microscopic torpedos. I fear some of these find an avowal of doubt the readiest means of excusing their ignorance; but where such doubts have been honestly entertained, I have never found them capable of withstanding the proof afforded by ruled-glass micrometers when placed under the microscope. Here, we know beforehand the angles at which the ruled lines intersect one another, and the geometric forms of the spaces they enclose; we can at once ascertain whether the microscope disturbs such angles, or whether it faithfully transmits them to the eye. If the latter, it is equally to be trusted in all its other demonstrations; and such we know to be the case. Other demurrers occasionally ask the question, Has not all been done in the way of discovery that can be done? is not the field exhausted? The absurdity of this objection to further inquiry is best shown by the fact that, as yet, we do not know the entire correct history of one single object! Many magnificent monographs have been written, and

none more remarkable than the classic one of Trembley "On the Fresh-water Polype," which, though penned a century ago, seemed to approach as nearly as any other to an exhaustion of the subject; but recent inquiries have shown us in how many points this history was incomplete. What a vast field do these and similar facts open to you. How many great subjects remain to be developed in Physiology — especially as bearing on the healing art, in Chemistry, in Geology, and, most of all, in the endless themes which Natural History brings before us!

You cannot go wrong in your search for subjects of study. The rocky crags of the everlasting hills abound with minute memorials of former life. The desert sands of Egypt and the rice-swamps of Carolina tell alike of ocean waves that once rolled over their spreading plains. The dusty ash, cast from the volcano's mouth and floating away upon the passing breeze, reveals organic atoms drawn from ocean depths. The minutest of the innumerable insects that dance in the laughing sunbeam display beauteous forms and elaborate structures. Earth, air, fire and water cast their wondrous treasures into the lap of the microscopist, and invite him to an exercise of his observative skill.

I must apologize for the rambling nature of this discourse; but I am anxious in every way to confirm and encourage you in studies of the fruits of which you cannot be robbed. They will abide with you in life and, I believe, will survive death. Of how few sources of earthly pleasure can this be said! *"Tædet me vitæ"* is an old cry: the feelings that prompted it were experienced long before the exiled Roman orator thus gave utterance to disappointed hopes and ungratified ambition; and it is still the cry of thousands wearied with *ennui* and wanting interest in the world around. But let our hearts first be right, whilst our heads and hands are occupied in cultiva-

ting those fields of research that we know to be so boundless, and the cry of "*Tædet me vitæ*" will never be heard from any of us.

That great intellectual results may crown your labours is my earnest wish. Though I would not teach that intellect affords any title to eternal life, I cannot believe that, other things being equal, the giant mind of a Newton and that of a clown will occupy the same level in the great day of account. That a man who, besides being a good and faithful follower of his Lord, has fostered his intellect, benefiting his fellow men and glorifying his God, will not find such culture influencing his condition in a future world, constitutes no part of my religious creed.

The first Paper of this Session, being the first read before the Section, was one by Mr. Geo. Mosley, *on "Daphnia Pulex," March 21st,* 1859.

Mr. Mosley commenced his account of the animal by enumerating some of its principal historians and describers from the year 1669 to the recent account of it by Baird, published in the Ray Society's volume for 1850. After giving descriptions of the animal, as it appears under 50 and 200 linear magnifying power, Mr. Mosley proceeded specially to examine a doubtful organ on the head of this crustacean.

"Proceeding," he says, "from the upper frontal extremity, a nerve about $\frac{1}{600}$ of an inch in length is produced to a small oval black spot $\frac{1}{2000}$ of an inch in its long diameter, situated between the eye and the beak. Baird does not name this organ in the Daphnia. In the Microthrix it is so much developed that he marks it as characteristic: 'Eye accompanied with a black spot.' This black spot was noticed by Müller, who considered it to be another eye. Jurine differs from this opinion, though unable to discover its real utility; so does Strauss, who found it in the embryo before birth, exactly as in the adult. Baird agrees with these last authors, that it is not an organ of vision. Dr. Zenker, in the *Microscopical Magazine* for 1850, vol. ii. second series, says it has been looked upon as an auditory organ, but compares it with the tripartite azygos eye, which occurs extensively in the crustaceans; but he leaves the question much as he found it. He says it is the first-developed organ of sense, and, reasoning from analogy

with the laws of development of other Branchiopoda, that it may be an eye in the embryonic state. It is evidently of great importance to the animal; and with all due deference to these distinguished authors (the suggestion is made with diffidence) may it not possibly be the organ of smell? Schödler's supposition that it is the ear, and Müller's that it is another eye, do not bear the impress of probability; whereas in most animals the sense of smell is situated between the eye and the mouth. Zenker confesses that an eye in the embryo could be of but little use; whereas a highly developed olfactory might be so. Again: immediately behind the eye may be observed several round spots each about $\frac{1}{1000}$ of an inch in diameter, from which a branched nerve extends across the nerves and muscles of the eye to the front of the larger lobe of the brain. These do not appear to have been noticed by any of the said writers; their use is unknown; but from their position in the head, and their connection with the brain, it is not unreasonable to suppose that they may form the true auditory organs. If these suggestions be correct, the organs of hearing and smelling will be placed behind and in front of the eye, in accordance with the laws which prevail in comparative anatomy."

After describing the male Daphnia, which Mr. Mosley says is not often to be met with, he concluded this paper by a lengthy description of the singular method of the reproduction of the animalcule.

At the meeting of the Section, April 18th, 1859, Mr. H. A. Hurst *read a Paper on the "Structure of Starch Granules."*

Mr. Hurst gives a long series of quotations from the works of the most celebrated writers on this subject, illustrative of the various opinions entertained by high authorities on the Structure of these Granules, showing the prominent opinion on the subject to be, that they owe the peculiar striated appearance of their surfaces to their being formed of a number of concentric layers converging to a nucleus. This theory, however, Mr. Hurst disputes; and after giving a detailed account of the manner in which the granules behave under the influence of various chemical reagents, proceeds to support a theory of his own on the subject as follows:

"Payen, in his *Mémoires sur le développement des Végétaux,* in 1842, gives plates of the appearances of Starch Granules, which on first inspection appear to prove satisfactorily the concentric layer theory; but on further examination it will be found that they require to be roasted, iodized, acidulated, and otherwise acted on by chemical agency to such an extent that the results cannot be accepted as evidence of structure existing before all these influences were brought to bear upon them; and in connection with this I would remark that immersing an object for three weeks in iodine, as done by Dr. Allman, is hardly a fair test, and may very possibly cause the concentric layers, to prove the existence of which it is used. Mr. Tuffen West remarks, and I consider very justly, that

appearances presented by, more especially, vegetable structure, after having undergone prolonged torture by chemical agency, are no more to be depended upon than those presented by a human being under similar circumstances.

"The striæ, if they were the optical expression of lines of separation of concentric layers, would, I submit, show strongest when the plane of the focus of the object glass coincided with the central plane of the granule; but this appears to me not to be the case. On the contrary, the striæ are only shown when the surface of the granule is in focus. Some starch grains do not show striæ at all, whilst those of the Canna Indica show them only on three sides.

"But a strong argument against the concentric layer theory is that the nucleus, where all the supposed layers are superimposed one over the other, is actually the most transparent part of the granule, instead of being the most opaque, which would be the natural consequence of the presence of a larger number of layers than those existing at the opposite end of the granule.

"This extreme end, where there can be only one layer, is, if there be any difference, the darkest part of the granule, as can be verified by inspection.

"In examining the action of any mineral acid, or of an alkali, on starch granules — say the largest — I find the following phenomena:

"Should the agent be much concentrated, the granule bursts at once the moment it is touched by it, and lies on the glass slide a flat ovate single membrane, no trace being discernible of striæ or internal layers. If more dilute, a gradual increase of size takes place, one or more of the striæ becoming strongly marked. The hilum or nucleus seems to be an aperture through which the agent obtains access to the interior, and from it stellate fissures arise, caused apparently by the absorption of the agent

and consequent expansion of the outer cell-wall or membrane; these gradually increase in length, exposing the interior to view, and allowing the granule to expand laterally till they almost reach the edge of the greatly enlarged granule, when, of course, the band of the cell, which has still retained its cellular or enclosing character, gives way, and the granule assumes the aspect previously described as immediately resulting from the application of a strong alkali or mineral acid. It is worthy of notice that, while these fissures are extending, the strip of membrane between any two, still retains the striæ, which are clearly visible. Now these fissures are, undoubtedly, not in the body of the grain, but are merely fissures of the outer layer or investing membrane; at least they appear to me to be so; and if such should be the case, I would suggest, it is a positive proof that these striæ do not arise from the existence of concentric layers, nor even from an unfolding of the outer layer, since they appear even if it has been rent by distention.

"That this exterior membrane or investing layer is of a different nature from the contents it encloses, I think no one can doubt who has observed the above described phenomena; but from what Dr. Gregory says, it would appear to be merely a difference of structure or texture, not of chemical composition; and after a careful consideration of my experiments, I am inclined to attribute the striæ to the circular deposition of rings of starch on the inner surface in precisely a similar manner to the deposition of circular or spiral rings of woody fibre on the sides of the so-called annular and spiral ducts in the tissue of many plants, more particularly those belonging to the natural family of Cucurbitaciæ, Orchidiæ and Filices. Drawings of the cells found in the leaves of species of Oncidium might easily be taken for representations of starch granules, so exact is the resemblance.

"This view derives confirmation from the fact, that the deeper marking of some striæ, when the granule is under the influence of acids or alkalies, is evidently caused, not by its own increase in depth or thickness, but by the expansion of the membrane in which it is deposited, each side clearly separating it further from the adjacent striæ; while the moment the extension of stellate fissures allows lateral expansion of the granule no further deepening of striæ is observable.

"I therefore consider the starch granule to be constituted of an enveloping membrane, or shell, of very firmly consolidated starch, enclosing a certain amount of the same substance in an amorphous condition and a more lax state, while the hilum, or nucleus, would, according to my view, be merely an aperture."

At the conclusion of this Paper some discussion took place as to the effect of various chemical reagents on the starch granule, and also as to the possibility of producing sections of them. The President suggested that the same method might be applied to the starch granule as was used to obtain sections of hairs, &c., that of mixing the granules with glue or Canada balsam, and, when hard, grinding the mass down to a thin layer.

SESSION, 1857-1858.

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1850	April 30.	Johnson, Richard, F.C.S.
1856	January 22.	Johnson, William.
1836	January 22.	Jones, Rev. Henry Halford, F.R.A.S.
1821	October 19.	Jordan, Joseph.
1848	April 18.	Joule, B. St. J. B.
1842	January 25.	Joule, James Prescott, LL.D., F.R.S., Hon. M.C.P.S., Corr. Mem. Roy. Acad. Sc. Turin.
1846	January 27.	Joynson, William.
1843	January 24.	Kay, Samuel.
1852	January 27.	Kennedy, John Lawson.
1854	January 24.	Kershaw, James, jun.
1822	April 26.	Lane, Richard.
1830	April 30.	Langton, William.
1850	April 30.	Leese, Joseph, jun.
1856	January 22.	Lingard, Thomas Taylor.
1825	January 27.	Littler, Thomas.
1839	October 29.	Lockett, Joseph.
1852	January 27.	Long, Isaac Waithman, F.R.A.S.
1857	January 27.	Longridge, R. B.
1842	April 19.	Love, Benjamin.
1854	January 24.	Lowe, George Cliffe.
1850	April 30.	Lund, Edward.
1855	January 23.	Lund, George T.
1855	October 30.	Mabley, William Tudor.
1829	October 30.	M'Connel, James.
1838	April 17.	M'Connel, William.
1844	April 30.	M'Dougall, Alexander.

Date of Election.	
1823	January 24. Macfarlane, John.
1849	April 17. Manchester, The Right Rev. the Lord Bishop of, D.D., F.R.S., F.G.S., F.C.P.S., Corr. Mem. Arch. Inst. Rome.
1858	April 20. Mather, Colin.
1842	January 25. Mellor, Thomas.
1837	January 27. Mellor, William.
1849	January 23. Morris, David.
1848	January 25. Neild, Alfred.
1822	April 26. Neild, William.
1852	January 27. Nelson, James Emanuel.
1854	February 7. Nevill, Thomas Henry.
1845	January 21. Nicholls, John Ashton, F.R.A.S.
1827	January 26. Nicholson, William.
1844	April 30. Ormerod, Henry Mere.
1844	April 30. Parr, George.
1833	April 26. Parry, John.
1841	April 15. Peel, George, M. Inst. C.E.
1857	April 21. Platt, William Wilkinson.
1854	January 24. Pochin, Henry Davis.
1857	January 27. Poynting, Rev. T. E.
1854	February 7. Ramsbottom, John.
1836	April 29. Ransome, Joseph Atkinson, F.R.C.S.
1847	January 26. Ransome, Thomas.
1823	January 18. Roberts, Richard, M. Inst. C.E.
1822	January 25. Robinson, Samuel.
1858	January 26. Roscoe, Henry Enfield, B.A., Ph.D., F.C.S., Professor of Chemistry, Owens College.
1842	January 25. Royle, Alan.
1848	April 18. Salt, Samuel.
1851	April 29. Sandeman, Archibald, M.A., Professor of Mathematics, Owens College.

Date of Election.	
1847	January 26. Satterthwaite, Michael, M.D.
1842	January 25. Schunck, Edward, Ph.D., F.R.S., F.C.S.
1854	February 7. Scott, Alexander J., M.A., Prof. of Logic and English Language, Owens College.
1855	January 23. Sharp, Edmund Hamilton.
1835	October 30. Shuttleworth, John.
1852	April 20. Sidebotham, Joseph.
1838	January 26. Smith, George S. Fereday, M.A., F.G.S.
1845	April 29. Smith, Robert Angus, Ph.D., F.R.S., F.C.S.
1851	April 29. Spence, Peter.
1852	January 27. Standring, Thomas.
1834	January 24. Stephens, Edward, M.D.
1847	April 20. Stephens, James.
1858	January 26. Stewart, Charles P.
1814	January 21. Stuart, Robert.
1856	January 22. Taylor, John Edward.
1852	April 20. Thom, David.
1836	April 29. Turner, James Aspinall, M.P.
1821	April 19. Turner, Thomas, F.R.C.S.
1857	January 27. Walker, Robert, M.D.
1823	January 24. Watkin, Absalom.
1857	January 27. Webb, T. G.
1858	January 26. Whitehead, James, M.D.
1832	January 22. Whitworth, Joseph, F.R.S.
1853	April 19. Williamson, Samuel Walker.
1851	April 29. Williamson, William Crawford, F.R.S., Professor of Natural History, Owens College.
1851	January 21. Withington, George Bancroft.
1839	January 22. Wood, William Rayner.
1855	October 30. Woodcock, Alonzo Buonaparte.
1846	April 21. Woodhead, George.
1839	April 30. Woods, Edward.
1840	April 28. Worthington, Robert, F.R.A.S.

CORRESPONDING MEMBERS.

Dockray, Benjamin, *Lancaster.*
Girardin, M. J., *Rouen.*
Harley, Rev. Robert, F.R.A.S., *Brighouse, Yorkshire.*
Kenrick, Rev. John, M.A., *York.*
Koechlin-Schouch, Daniel, *Mulhouse, France.*
Lowe, E. J., F.R.A.S., *Nottingham.*
Mercer, John, F.R.S., *Oakenshaw, Accrington, Lancashire.*
Pincoffs, Peter, M.D., *Dresden.*
Watson, Henry Hough, *Bolton, Lancashire.*
Wilkinson, T. T., F.R.A.S., *Burnley, Lancashire.*

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THE

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APRIL 19, 1859.

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PETER SPENCE.
JOSEPH ATKINSON RANSOME, *M.R.C.S.*

HONORARY MEMBERS.

DATE OF ELECTION.

1847, Apr. 20. Adams, John Couch, F.R.S., F.R.A.S., F.C.P.S., Mem. Amer. Acad. Arts and Sc. Boston. *Pembroke College, Cambridge.*

1843, Apr. 18. Agassiz, Louis, F.R.S., Mem. Imper. Instit. France, &c. *Cambridge, Massachusets.*

1843, Feb. 7. Airy, George Biddell, M.A., D.C.L., F.R.S., Astronomer Royal, V.P.R.A.S., Hon. Mem. R.S.E., R.I.A., M.C.P.S., Chev. of the Prussian Order "*Pour le Mérite*," Corr. Mem. Nat. Inst. Washington, U.S., Imper. Instit. France, Imper. Acad. Sc. Petersburg and Roy. Acad. Sc. Berlin, Mem. Acad. Sc. and Lit. Palermo, Roy. Acadd. Sc. Stockholm and Munich, Roy. Soc. Sc. Copenhagen, and Amer. Acad. Arts and Sc. Boston. *The Royal Observatory, Greenwich, London, S.E.*

1843, Apr. 18. Barlow, Peter, F.R.S., F.R.A.S., Hon. M.C.P.S., Mem. Imper. Acad. Sc. Petersburg, and Amer. Acad. Arts and Sc. Boston, Corr. Mem. Imper. Instit. France and Roy. Acad. Sc. Brussels. *Woolwich, London, S.E.*

1849, Jan. 23. Bosworth, Rev. Joseph, LL.D., F.R.S., F.S.A., M.R.I.A., Corr. Mem, Roy. Soc. Northern Antiq. Copenhagen, Mem. Roy. Soc. Sc. Drontheim and Soc. Sc. Gothenburg. *Islip House, near Oxford, and Etwell, Uttoxeter.*

1843, Apr. 18. Brewster, Sir David, K.H., LL.D., F.R.S.S. L. & E., Hon. M.R.I.A., F.G.S., F.R.A.S., Chev. of the Prussian Order "*Pour le Mérite*," Mem. Imper. Instit. France and Roy. Soc. Sc. Gottingen, Principal of St. Leonard's College. *St. Andrews.*

DATE OF ELECTION.

1843, Feb. 7. Brisbane, General Sir Thomas Makdougall, Bart., K.C.B., G.C.H., D.C.L., F.R.S., Pres. R.S.E., Hon. M.R.I.A., F.R.A.S., F.L.S., Corr. Mem. Imper. Institut. France. *Makerstoun, Kelso.*

1859, Jan. 25. Cayley, Arthur, M.A., F.R.S., F.R.A.S. 2, *Stone Buildings, Lincoln's Inn, London, W.C.*

1859, Jan. 25. De Morgan, Augustus, V.P.R.A.S., F.C.P.S., Professor of Mathematics in Univ. Coll. London. 41, *Chalcot Villa, Adelaide Road, London.*

1844, Apr. 30. Dumas, Jean Baptiste, F.R.S., Mem. Imper. Institut. France, &c. *Paris.*

1843, Apr. 18. Faraday, Michael, D.C.L., F.R.S., Fullerian Prof. of Chemistry in the Roy. Institut. of Great Britain, Hon. Mem. R.S.E., C.P.S., and Med. Chir. Soc., F.G.S., Chev. of the Prussian Order "*Pour le Mérite,*" Comm. Legion of Honour, Mem. Imper. Institut. France, Imper. Acadd. Sc. Vienna and Petersburg, Roy. Acadd. Sc. Berlin, Turin, Stockholm, Munich, Naples, Amsterdam, Brussels, Bologna and Modena, Roy. Socc. Sc. Gottingen, Copenhagen, Upsal and Harlem, Amer. Acad. Arts and Sc. Boston, and Amer. Phil. Soc. Philadelphia, Corr. Mem. Acad. Sc. and Liter. Palermo, Acad. Georg. Florence, Philom. Soc. Paris, Nat. Institut. Washington U.S., and Imper. Acad. of Med. Paris, &c. 21, *Albemarle Street, London, W.*

1843, Feb. 7. Frisiani, nobile Paolo, Prof., 2nd Astron. at the Observ. of Brera, Milan, Mem. Imper. Roy. Institut. of Lombardy, Milan, and Ital. Soc. Sc. *Milan.*

1843, Feb. 7. Hamilton, Sir William Rowan, Knt., LL.D.,

DATE OF ELECTION.

M.R.I.A., F.R.A.S., Astronomer Royal, Ireland, Andrews Professor of Astronomy, T.C.D. *Observatory, Dunsink, Dublin.*

1843, Feb. 7. Harcourt, Rev. William Venables Vernon, M.A., F.R.S., Hon. M.R.I.A., F.G.S. *Bolton Percy, Tadcaster.*

1853, Apr. 19. Hartnup, John, F.R.A.S. *Observatory, Liverpool.*

1843, Apr. 18. Herschel, Sir John Frederick William, Bart., D.C.L., F.R.S.S. L. & E., Hon. M.R.I.A., F.G.S., M.C.P.S., Chev. of the Prussian Order "*Pour le Mérite,*" Corr. Mem. Imper. Instit. France, Mem. Imper. Acad. Sc. Petersburg, Roy. Acadd. Sc. Berlin, Turin, Naples and Brussels, Roy. Socc. Sc. Gottingen, Copenhagen and Harlem, Acad. dei Nuovi Lincei Rome, Acadd. Padua, Bologna, Palermo, Modena, Acad. Gioen. Catania, Imper. Acad. Sc. Dijon, Philom. Soc. Paris, and Soc. Nat. Sc. Switzerland. *Collingwood, near Hawkhurst, Kent.*

1848, Jan. 25. Hind, John Russell, F.R.A.S. *Regent's Park, London, N.W.*

1853, Jan. 25. Hopkins, William, M.A., LL.D., F.R.S., F.G.S. *Fitzwilliam Street, Cambridge.*

1852, Oct. 19. Kirkman, Rev. Thomas Penynton, M.A., F.R.S. *Croft Rectory, near Warrington.*

1848, Oct. 31. Lassell, William, F.R.S., F.R.A.S., Hon. F.R.S.E., Hon. M. Philomath Soc. Paris. *Bradstones, Sandfield Park, near Liverpool.*

1847, Apr. 20. Le Verrier, Urbain Jean Joseph, F.R.S., Mem. Imper. Instit. France, &c. *Paris.*

1843, Feb. 7. Liebig, Justus Baron von, M.D., Ph.D., Prof. of Chem. Univ. Munich, Conservator of Chem. Labor. Munich, Chev. of the Bav. Order "*Pour le Mérite,*" &c., F.R.S.S. L. & E., Hon.

DATE OF ELECTION.

M.R.I.A., Hon. Mem. Univ. Dorpat and Med. Phis. Facult. Univ. Prague, Hon. Mem. and For. Assoc. Roy. Acad. Sc. Vienna, Roy. Acadd. Sc. Stockholm, Brussels, Amsterdam, Turin, Acad. Sc. Bologna, Roy. Socc. Sc. Gothenburg, Gottingen, Copenhagen, Liege, Imper. Roy. Instit. of Lombardy, Milan, Corr. Mem. Imper. Instit. France, Imper. Acad. Sc. Petersburg, Roy. Acad. Sc. Madrid, Mem. Roy. Med. Chir. Soc. London and Perth, Roy. Scot. Soc. Arts, Botan. Socc. Edinburgh and Regensburg, Socc. Nat. Sc. Berlin, Dresden, Halle, Moscow, Lille, Ph. Soc. Glasgow, Agric. Socc. Munich, Giessen, &c. &c. *Munich.*

1849, Apr. 17. Mercer, John, F.R.S. *Oakenshaw, Accrington, Lancashire.*

1843, Feb. 7. Mitscherlich, Eilert, Professor, F.R.S., &c. *Berlin.*

1854, Jan. 24. Morin, Arthur, General of Artillery, Mem. Imper. Instit. France, formerly élève Polytechn. School, Dir. Conserv. of Arts, Paris, Corr. Mem. Roy. Acadd. Sc. Berlin, Madrid and Turin, Acad. Georg. Florence, Imper. Acad. Metz, and Industr. Soc. Mulhouse. *Paris.*

1843, Apr. 18. Moseley, Rev. Henry, M.A., F.R.S., Corr. Mem. Imper. Instit. France. *Olveston, near Bristol.*

1821, Jan. 26. Mosley, Sir Oswald, Bart. *Rolleston Hall.*

1844, Apr. 30. Murchison, Sir Roderick Impey, G.C.St.S., D.C.L., M.A., F.R.S., F.G.S., F.L.S., &c., Director Gen. of the Geol. Survey, Pr. R.G.S., Hon. Mem. R.S.E., and R.I.A., Mem. C.P.S. and Imper. Acad. Sc. Petersburg, Corr. Mem. Imper. Instit. France, Roy. Acadd. Sc. Stockholm, Turin, Berlin and Brussels, Roy. Soc. Sc. Copenhagen, Amer. Acad. Arts and Sc. Boston, and Imper. Geogr. Soc. Petersburg, Hon. Mem. Imper. Soc. of Naturalists, Moscow, &c. &c. 16, *Belgrave Square, London, S.W.*

DATE OF ELECTION.

1844, Apr. 30. Owen, Richard, M.D., LL.D., V.P.R.S., F.L.S, F.G.S., V.P.Z.S., Director of the Nat. Hist. Department, British Museum, Hon. F.R.C.S. Ireland, Hon. M.R.S.E., For. Assoc. Imper. Instit. France, Mem. Imper. Acadd. Sc. Vienna and Petersburg, Imper. Soc. of Naturalists Moscow, Royal Acadd. Sc. Berlin, Turin, Madrid, Stockholm, Munich, Amsterdam, Naples, Brussels and Bologna, Roy. Socc. Sc. Copenhagen and Upsal, and Amer. Acad. Arts and Sc. Boston, Corr. Mem. Philom. Soc. Paris. Mem. Acad. Georg. Florence, Soc. Sc. Harlem and Utrecht, Soc. of Phys. and Nat. Hist. Geneva, Acad. dei Nuovi Lincei Rome, Roy. Acadd. Sc. Padua, Palermo, Acad. Gioen. Catania, Phys. Soc. Berlin, Chev. of the Prussian Order "*Pour le Mérite*," For. Assoc. Instit. Wetter., Philadelphia, New York, Boston, Imper. Acad. Med. Paris, and Imper. and Roy. Med. Soc. Vienna. *British Museum, London, W.C.*

1851, Apr. 29. Playfair, Lyon, C.B., Ph.D., F.R.S., F.G.S., F.C.S., Professor of Chemistry Univ. Ed. *Edinburgh.*

1856, Jan. 22. Poncelet, General Jean Victor, F.R.S., Mem. Imper. Instit. France, &c. *Paris.*

1859, Apr. 19. Rankine, William John Macquorn, LL.D., F.R.S.S. L. & E., Pres. Inst. Eng. Scot., Regius Professor of Civil Engineering and Mechanics, Univ. Glasgow. *59, St. Vincent Street, Glasgow.*

1849, Jan. 23. Rawson, Robert. *Royal Dockyard, Portsmouth.*

1859, Apr. 19. Reichenbach, Carl Baron von. *Vienna.*

1844, Apr. 30. Sabine, Major-General Edward, R.A., D.C.L., Treas. and V.P.R.S., F.R.A.S., Hon. Mem.

DATE OF ELECTION.

C.P.S., Chev. of the Prussian Order "*Pour le Mérite*," Mem. Imper. Acad. Sc. Petersburg, Roy. Acadd. Sc. Berlin, Brussels and Gottingen, Roy. Soc. Sc. Drontheim, Acad. Sc. Philadelphia, Econ. Soc. Silesia, Nat. Hist. Soc. Lausanne and Roy. Batavian Soc., Corr. Mem. Roy. Acad. Sc. Turin, Nat. Instit. Washington U.S., Geogr. Socc. Paris, Berlin and Petersburg. 13, *Ashley Place, Westminster, London, S.W.*

1843, Feb. 7. Sedgwick, Rev. Adam, M.A., F.R.S., Hon. M.R.I.A., F.G.S., F.R.A.S., Woodwardian Lecturer. *Cambridge.*

1851, Apr. 29. Stokes, George Gabriel, M.A., D.C.L., Secr. R.S., Lucasian Professor of Mathem., F.C.P.S., Mem. Batav. Soc. Rotterdam. *Pembroke College, Cambridge.*

1854, Jan. 24. Tayler, Rev. John James, B.A., Principal of Manchester New College. *London.*

1851, Apr. 29. Thomson, William, M.A., LL.D., F.R.S.S. L. & E., Professor of Nat. Philos. Univ. Glasgow. 2, *College, Glasgow.*

1843, Feb. 7. Whewell, Rev. William, D.D., F.R.S., Hon. M.R.I.A., F.S.A., F.G.S., F.R.A.S., Master of Trinity College, Cambridge. *Lodge, Cambridge.*

1850, Apr. 30. Woodcroft, Bennet, Professor, Superint. of Regist. of Patents. *Southampton Buildings, London, W.C.*

ORDINARY MEMBERS.

Date of Election.	
1857, Jan. 27.	Acton, Henry Morell, B.A.
1856, Apr. 29.	Adshead, Joseph.
1839, Apr. 30.	Ainsworth, Ralph Fawsett, M.D.
1854, Jan. 24.	Allan, James, Ph.D., M.A.
1837, Aug. 11.	Ashton, Thomas.
1846, Jan. 27.	Atkinson, John.
1824, Jan. 23.	Barbour, Robert.
1852, Jan. 27.	Barlow, Henry Bernoulli.
1842, Apr. 19.	Barratt, Joseph.
1849, Apr. 17.	Bassnett, Rev. Richard, M.A.
1840, Jan. 21.	Bateman, John Frederick, M.Inst. C.E., F.G.S.
1858, Jan. 26.	Baxendell, Joseph, F.R.A.S.
1847, Jan. 26.	Bazley, Thomas, M.P.
1847, Jan. 26.	Bell, William.
1857, Apr. 21.	Bellhouse, Edward Taylor.
1858, Jan. 26.	Benson, Davis.
1844, Jan. 23.	Bevan, James.
1854, Jan. 24.	Beyer, Charles.
1842, Jan. 25.	Binney, Edward William, F.R.S., F.G.S.
1821, Jan. 26.	Blackwall, John, F.L.S.
1855, Jan. 23.	Bowman, Eddowes, M.A.
1839, Oct. 29.	Bowman, Henry.
1859, Jan. 25.	Brittain, Thomas.
1857, Apr. 21.	Brock, John.
1855, Apr. 17.	Brockbank, William.
1824, Apr. 30.	Brooke, Edward.
1844, Jan. 23.	Brooks, William Cunliffe, M.A.
1846, Jan. 27.	Browne, Henry, M.B.
1846, Jan. 27.	Burd, John.
1854, Jan. 24.	Callender, William, jun.

DATE OF ELECTION.

1847, Jan. 26. Calvert, Frederick Crace, Ph.D., F.R.S., F.C.S., Corr. Mem. Roy. Acad. Sc. Turin, Acad. Sc. Rouen, Pharmac. Soc. Paris and Industr. Soc. Mulhouse.
1859, Jan. 25. Carrick, Thomas.
1858, Jan. 26. Casartelli, Joseph.
1855, Jan. 23. Cawley, Charles Edward, M. Inst. C.E.
1852, Apr. 20. Chadwick, David, F.S.S., Assoc. Inst. C.E.
1842, Jan. 25. Charlewood, Henry.
1854, Apr. 18. Christie, Richard Copley, M.A., Prof. Hist. Owens College.
1857, Apr. 21. Churchill, George Cheetham.
1841, Apr. 20. Clay, Charles, M.D.
1853, Apr. 19. Clift, Samuel, F.C.S.
1853, Jan. 25. Corbett, Edward.
1853, Jan. 25. Cottam, Samuel.
1859, Jan. 25. Coward, Edward.
1851, Apr. 29. Crompton, Samuel.
1839, Jan. 22. Crossley, James, F.S.A., Pres. Chet. Soc.
1848, Jan. 25. Crowther, Joseph Stretch.
1843, Apr. 18. Curtis, Matthew.

1854, Feb. 7. Dale, John.
1842, Apr. 19. Dancer, John Benjamin, F.R.A.S.
1853, Apr. 19. Darbishire, Robert Dukinfield.
1854, Jan. 24. Davies, David Reynold.
1842, Nov. 15. Dean, James Joseph.
1855, Jan. 23. Dickinson, William Leeson.
1859, Jan. 25. Dorrington, James.
1850, Apr. 30. Dyer, Frederick Nathaniel.
1818, Apr. 24. Dyer, Joseph Chesborough.

1859, Jan. 25. Eadson, Richard.
1856, Apr. 29. Ekman, Charles Frederick.
1854, Jan. 24. Ellis, Charles.

1850, Apr. 30. Fairbairn, Thomas.

DATE OF ELECTION.

1824, Oct. 29. Fairbairn, William, C.E., F.R.S., F.G.S., Corr. Mem. Imper. Inst. France and Roy. Acad. Sc. Turin.
1849, Oct. 30. Fairbairn, William Andrew.
1842, Jan. 25. Fleming, David Gibson.
1817, Oct. 31. Flint, Richard.
1856, Apr. 29. Forrest, Henry Robert.
1857, Apr. 21. Foster, Thomas Barham.
1855, Jan. 23. Fothergill, Benjamin.
1854, Jan. 24. Fryer, Alfred.

1840, Jan. 21. Gaskell, Rev. William, M.A.
1836, Apr. 20. Giles, Samuel.
1847, Apr. 20. Gould, John.
1837, Aug. 11. Graham, John.
1817, Jan. 24. Greg, Robert Hyde, F.G.S.
1849, Oct. 30. Greg, Robert Philips, F.G.S.
1848, Jan. 25. Grundy, John Clowes.

1844, Jan. 23. Hampson, Richard.
1858, Oct. 19. Harrison, William Philip, M.D.
1839, Jan. 22. Hawkshaw, John, F.R.S., F.G.S., M. Inst. C.E.
1859, Apr. 19. Heelis, Thomas.
1828, Oct. 31. Henry, William Charles, M.D., F.R.S.
1854, Jan. 24. Hetherington, John.
1815, Jan. 27. Heywood, Sir Benjamin, Bart., F.R.S.
1833, Apr. 26. Heywood, James, F.R.S., F.G.S., F.S.A.
1851, Apr. 29. Higgin, James.
1845, Apr. 29. Higgins, James.
1848, Oct. 31. Higson, Peter.
1839, Jan. 22. Hobson, John.
1820, Jan. 21. Hodgkinson, Eaton, F.R.S., F.G.S., M.R.I.A., Hon. Mem. R.I.B.A. Inst. C.E., Roy. Scot. Soc. Arts, and Soc. Civ. Eng. Paris, Prof. of the Mech. Princ. of Engineering Univ. Coll. London.
1854, Jan. 24. Holcroft, George.

Date of Election.	
1855, Jan. 23.	Holden, Isaac.
1846, Jan. 27.	Holden, James Platt.
1823, Apr. 18.	Hopkins, Thomas, M. Brit. Met. Soc.
1824, Jan. 23.	Houldsworth, Henry.
1857, Jan. 27.	Hunt, Edward, B.A., F.C.S.
1859, Jan. 25.	Hurst, Henry Alexander.
1857, Jan. 27.	Jackson, George.
1823, Jan. 24.	Jesse, John, F.R.S., F.R.A.S., F.L.S.
1850, Apr. 30.	Johnson, Richard, F.C.S.
1856, Jan. 22.	Johnson, William.
1821, Oct. 19.	Jordan, Joseph.
1848, Apr. 18.	Joule, Benjamin St. John Baptist.
1842, Jan. 25.	Joule, James Prescott, LL.D., F.R.S., F.C.S., Hon. M.C.P.S., Corr. Mem. Roy. Acad. Sc. Turin.
1846, Jan. 27.	Joynson, William.
1843, Jan. 24.	Kay, Samuel.
1852, Jan. 27.	Kennedy, John Lawson.
1822, Apr. 26.	Lane, Richard.
1830, Apr. 30.	Langton, William.
1850, Apr. 30.	Leese, Joseph, jun.
1856, Jan. 22.	Lingard, Thomas Taylor.
1839, Oct. 29.	Lockett, Joseph.
1852, Jan. 27.	Long, Isaac Waithman, F.R.A.S.
1857, Jan. 27.	Longridge, Robert Bentink.
1842, Apr. 19.	Love, Benjamin.
1854, Jan. 24.	Lowe, George Cliffe.
1850, Apr. 30.	Lund, Edward.
1855, Jan. 23.	Lund, George Taylor.
1859, Jan. 25.	Lynde, James George, M. Inst. C.E., F.G.S.
1855, Oct. 30.	Mabley, William Tudor.
1829, Oct. 30.	McConnel, James.
1838, Apr. 17.	McConnel, William.

DATE OF ELECTION.

1844, Apr. 30. McDougall, Alexander.
1823, Jan. 24. Macfarlane, John.
1859, Jan. 25. Maclure, John William.
1849, Apr. 17. Manchester, The Right Rev. the Lord Bishop of, D.D., F.R.S., F.G.S., F.C.P.S., Corr. Mem. Arch. Inst. Rome.
1858, Apr. 20. Mather, Colin.
1842, Jan. 25. Mellor, Thomas.
1837, Jan. 27. Mellor, William.
1859, Jan. 25. Molesworth, Rev. William Nassau, M.A.
1849, Jan. 23. Morris, David.
1859, Jan. 25. Mosley, George.

1848, Jan. 25. Neild, Alfred.
1822, Apr. 26. Neild, William.
1852, Jan. 27. Nelson, James Emanuel.
1854, Feb. 7. Nevill, Thomas Henry.
1827, Jan. 26. Nicholson, William.

1844, Apr. 30. Ormerod, Henry Mere.

1844, Apr. 30. Parr, George.
1833, Apr. 26. Parry, John.
1841, Apr. 20. Peel, George, M. Inst. C.E.
1857, Apr. 21. Platt, William Wilkinson.
1854, Jan. 24. Pochin, Henry Davis.
1857, Jan. 27. Poynting, Rev. Thomas Elford.

1854, Feb. 7. Ramsbottom, John.
1859, Apr. 19. Ransome, Arthur, M.A., M.B. Cantab., M.R.C.S.
1836, Apr. 29. Ransome, Joseph Atkinson, F.R.C.S.
1847, Jan. 26. Ransome, Thomas.
1859, Jan. 25. Rideout, William Jackson.
1823, Apr. 18. Roberts, Richard, M. Inst. C.E.
1822, Jan. 25. Robinson, Samuel.
1858, Jan. 26. Roscoe, Henry Enfield, B.A., Ph.D., F.C.S., Professor of Chemistry, Owens College.

Date of Election.	
1842, Jan. 25.	Royle, Alan.
1851, Apr. 29.	Sandeman, Archibald, M.A., Professor of Mathematics, Owens College.
1847, Jan. 26.	Satterthwaite, Michael, M.D.
1842, Jan. 25.	Schunck, Edward, Ph.D., F.R.S., F.C.S.
1858, Oct. 19.	Sever, Charles.
1855, Jan. 23.	Sharp, Edmund Hamilton.
1835, Oct. 30.	Shuttleworth, John.
1852, Apr. 20.	Sidebotham, Joseph.
1859, Jan. 25.	Slagg, John, jun.
1838, Jan. 26.	Smith, George S. Fereday, M.A., F.G.S.
1845, Apr. 29.	Smith, Robert Angus, Ph.D., F.R.S., F.C.S., Corr. Mem. I. R. Geol. Instit. Vienna.
1859, Jan. 25.	Sowler, Thomas.
1851, Apr. 29.	Spence, Peter.
1852, Jan. 27.	Standring, Thomas.
1834, Jan. 24.	Stephens, Edward, M.D.
1847, Apr. 20.	Stephens, James.
1858, Jan. 26.	Stewart, Charles Patrick.
1814, Jan. 21.	Stuart, Robert.
1859, Jan. 25.	Tait, Mortimer Lavater.
1856, Jan. 22.	Taylor, John Edward.
1859, Jan. 25.	Thompson, James.
1836, Apr. 29.	Turner, James Aspinall, M.P.
1821, Apr. 19.	Turner, Thomas, F.R.C.S.
1857, Jan. 27.	Walker, Robert, M.D.
1823, Jan. 24.	Watkin, Absalom.
1859, Jan. 25.	Watson, John.
1857, Jan. 27.	Webb, Thomas George.
1858, Jan. 26.	Whitehead, James, M.D.
1839, Jan. 22.	Whitworth, Joseph, F.R.S.
1859, Jan. 25.	Wilde, Henry.
1859, Apr. 19.	Wilkinson, Thomas Read.
1853, Apr. 19.	Williamson, Samuel Walker.

DATE OF ELECTION.

1851, Apr. 29. Williamson, William Crawford, F.R.S., Professor of Natural History, Owens College.
1851, Jan. 21. Withington, George Bancroft.
1836, Jan. 22. Wood, William Rayner.
1855, Oct. 30. Woodcock, Alonzo Buonaparte.
1846, Apr. 21. Woodhead, George.
1839, Apr. 30. Woods, Edward.
1840, Apr. 28. Worthington, Robert, F.R.A.S.

CORRESPONDING MEMBERS.

1824, Jan. 23. Dockray, Benjamin, *Lancaster.*
1849, Apr. 17. Girardin, M. J., Corr. Mem. Imper. Instit. France, *Rouen.*
1850, Apr. 30. Harley, Rev. Robert, F.R.A.S., *Brighouse, Yorkshire.*
1816, Apr. 26. Kenrick, Rev. John, M.A., *York.*
1838, Apr. 17. Koechlin-Schouch, Daniel, *Mulhouse.*
1859, Jan. 25. Le Jolis, Augustus, Hon. Perpet. Archiviste Imper. Soc. Nat. Sc. Cherbourg, Mem. Imper. Leopold-Carol. Acad. Nat. Sc., *Cherbourg.*
1857, Jan. 27. Lowe, Edward Joseph, F.R.A.S., F.G.S., M.B.M.S., Hon. M. Dublin Nat. Hist. Soc., M. Geol. Soc. Edinb., &c., *Nottingham.*
1851, Apr. 29. Pincoffs, Peter, M.D., Chev. of the Turkish Order of the "*Medjidié*" 4th Cl., Mem. Coll. Phys. London, Brussels and Dresden, Hon. and Corr. Mem. Med. and Phil. Socc. Antwerp, Athens, Brussels, Constantinople, Dresden, Rotterdam, Vienna, &c., &c., *Constantinople.*

DATE OF ELECTION.

1834, Jan. 24. Watson, Henry Hough, *Bolton, Lancashire.*

1853, Apr. 19. Wilkinson, Thomas Turner, F.R.A.S., *Burnley, Lancashire.*

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